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Counting BPS States via Holomorphic Anomaly Equations

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Abstract. We study Gromov-Witten invariants of a rational elliptic surface using holomorphic anomaly equation in [HST1]. Formulating invariance under the affine E_8 Weyl group symmetry, we determine conjectured invariants, the number of BPS states, from Gromov-Witten invariants. We also connect our holomorphic anomaly equation to that found by Bershadsky,Cecotti,Ooguri and Vafa [BCOV1].

1 Introduction and Main results

Let S be a surface obtained by blowing up nine base points of two generic cubics in $\mathbf{P}^2$. S has an elliptic fibration $f : S \to \mathbf{P}^1$ and, in this note, we call it rational elliptic surface or 12K3. (The latter name comes from the fact that S has 12 singular fibers of Kodaira I_1 type while a generic elliptic K3 surface has 24.)

The surface S is of considerable interest in the study of Gromov-Witten invariants and, in fact, has been providing a testing ground for (local) mirror symmetry [KMV] of Calabi-Yau threefolds and its applications to enumerative geometry. For example, in [HSS] the celebrated Modell-Weil group of S has been connected to certain genus zero Gromov-Witten invariants of S. In [HST1], a certain recursion relation (*holomorphic anomaly equation*) was found, which determines the generating function of Gromov-Witten invariants of S for all genera. The main purpose of this note is to present a detailed study of the solutions of the holomorphic anomaly equation. Also we study Gromov-Witten invariants using similar but more general holomorphic anomaly equation valid for all Calabi-Yau threefolds due to [BCOV1,2], and remark a nontrivial relation between two equations. Main results in this paper are **Proposition 2.4**, **Tables 2–5**, and **Conjecture 4.3**.

To describe the setting in more detail, let us consider a Calabi-Yau threefold X which contain a rational elliptic surface S. Consider the moduli space of stable maps from genus g curves with n point on it to S. Then genus g Gromov-Witten

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invariant $N_g(\beta)$ with $\beta \in H_2(S, \mathbf{Z})$ is defined by

$$N_g(\beta) = \int_{[\bar{\mathcal{M}}_{g,0}(S,\beta)]^{vert}} c(R^1\pi_*\mu^* N_{S/X}) \ \ , \tag{1.1}$$

where $N_{S/X}$ is the normal sheaf and $\mu : \mathcal{M}_{g,1}(S,\beta) \to S$ is the evaluation map and $\pi : \mathcal{M}_{g,1}(S,\beta) \to \mathcal{M}_{g,0}(S,\beta)$ is the forgetful map.

For some special β, using localization method of torus actions, we may calculate $N_g(\beta)$ directly based on the definition (1.1), see e.g. [Ko][KZ] for details. Another way to determine $N_g(\beta)$ is to use the calculational technique based on mirror symmetry conjecture in [CdOGP] and [BCOV1]. Although the latter way has great advantage in calculating Gromov-Witten invariants, its equivalence to the abstract definition (1.1) has been established in [G] and [LLY1] only for some restricted Calabi-Yau hypersurfaces, see also [CK] for more backgrounds. Our holomorphic anomaly equation for S came from the calculational technique based on the mirror symmetry[HST1].

(1) To reproduce the holomorphic anomaly equation more specifically, let F and σ in $H^2(S,\mathbf{Z})$, respectively, be the fiber class and the class of a section of the elliptic fibration. Then consider the following summation over β;

$$N_g(d,n) := \sum_{\beta.\sigma=d,\ \beta.F=n} N_g(\beta)$$

and define the corresponding generating function with formal variable q;

$$Z_{g;n}(q) := \sum_{d\geq 0} N_g(d,n) q^d \ \ . \tag{1.2}$$

In [HST1], generalizing the result in [MNW] for $g=0$, it was found that:
(Holomorphic anomaly equation): The generating function $Z_{g;n}$ has the form

$$Z_{g;n}(q) = P_{g,n}(E_2(q), E_4(q), E_6(q)) q^{n2} \eta(q)^{12n} \tag{1.3}$$

with some quasi-modular form $P_{g,n} \in \mathbf{Q}[E_2,E_4,E_6]$ of weight $2g+6n-2$, where E_2,E_4,E_6 are Eisenstein series of weight two, four and six, respectively, and $\eta(q) = q^{124}\prod_{m>0}(1-q^m)$. Moreover $Z_{g;n}$ satisfies

$$\partial Z_{g;n} \partial E_2 = 124 \sum_{g'+g''=g} \sum_{s=1}^{n-1} s(n-s) Z_{g';s}\ Z_{g'';n-s} + n(n+1)24 Z_{g-1;n} \ , \tag{1.4}$$

with the initial data $Z_{0;1}(q) = q^{12}E_4(q)\eta(q)^{12}$.

One of the interesting features of this equation is that, under certain additional vanishing conditions (gap condition) on $N_g(d,n)$, we can determine $Z_{g;n}(q)$ for all $g \geq 0$ and $n \geq 1$. Some explicit formulas are presented in the end of this section. In this paper, using the affine E_8 Weyl symmetry which arises as isomorphisms of rational elliptic surfaces [Lo][Do], we will determine $N_g(\beta)$ for $\beta \in H_2(S,\mathbf{Z})$ with $(\beta,F) = n = 1,2,3,4$ and $g = \frac{1}{2}\{(\beta,\beta)-(\beta,F)+2\} \leq 10$. (Proposition 2.4 and Tables 2–5.)

(2) Another important aspect of Gromov-Witten invariants is that the invariants take values in $\mathbf{Q}$, however these can be related to *integer* "invariants" which, for example, may be identified with the number of (rational) curves in Calabi-Yau manifold. The relation to the integer "invariants" has appeared as multiple cover formula in [CdOGP] and [AM] for genus $g=0$, and its most general form has

been proposed by Gopakumar and Vafa giving physical meanings for the integer "invariants", i.e. the number of *BPS states*:
(Gopakumar-Vafa conjecture): Gromov-Witten invariants $N_g(\beta)$ are related to integer invariants $n_g(\beta)$ (the number of BPS states of charge β) by

$$N_g(\beta) = \sum_{k|\beta} \sum_{h=0}^{g} C(h, g-h) k^{2g-3} n_h(\beta/k) \ , \tag{1.5}$$

where $\sum_{k|\beta}$ means the summation over positive integer k which divide the integral class β, and $C(h, g-h)$ is the rational number defined by

$$(\sin(t/2)t/2)^{2g-2} = \sum_{h=0}^{\infty} C(g,h) t^{2h} \ .$$

Our result in this respect is that we verify the integrality of $n_g(\beta)$ up to $g \le 10$ and $\beta.F \le 4$ for rational elliptic surface S. (Tables 2–5.) Gopakumar and Vafa have also proposed that the integer "invariants" $n_g(\beta)$ should be geometric invariants on the moduli space of D2 branes of charge β, i.e. suitable moduli space of curves of a fixed homology class β and with local system on it. Precise mathematical definition of the moduli space of D2 branes $\mathcal{M}_\beta(X)$ has been proposed in [HST2] for Calabi-Yau threefold X with an ample class L. There the moduli space $\mathcal{M}_\beta(X)$ is defined as the normalization of the moduli space of semistable sheaves of pure dimension one with its support having homology class β, and also with a fixed Hilbert polynomial $P(m) = dm + 1$ ($d = L \cdot \beta$). Some numbers $n_g(\beta)$ have been explained from this definition[HST2]. We will provide a brief scketch in sect.3.3 about the expected geometrical interpretation about the numbers $n_g(\beta)$, although its detailed study is beyond the scope of our note. Here we remark that in case of elliptic surfaces, like $\frac{1}{2}$K3, the moduli spaces of D2 branes may be mapped to the moduli space of stable sheaves on the surface under fiberwise Fourier-Mukai transformations, see for example [MNVW], [Yo],[HST3].

(3) The most general form of the holomorphic anomaly equation which is applicable, in principle, to arbitrary Calabi-Yau threefold is known in [BCOV1,2]. We will connect our holomorphic anomaly equation (1.4) to a certain limit (local mirror symmetry limit) of the equations in [BCOV1,2]. We will make explicit comparisons of these two equations for $g = 2, 3$, and conjecture their equivalence. (Conjecture 4.3). Also we will find a nontrivial relation in the holomorphic ambiguities of these equations.

Finally, for reader's convenience, we present here some explicit forms of solutions of the holomorphic anomaly equation (1.4):

$$Z_{1,1}(q) = \frac{E_2(q)E_4(q)}{\prod_{n\ge1}(1-q^n)^{12}} \ , \quad Z_{2,1}(q) = \frac{E_4(q)(5E_2(q)^2 + E_4(q))}{1440\prod_{n\ge1}(1-q^n)^{12}}$$

$$Z_{3,1}(q) = \frac{E_4(q)(35E_2(q)^3 + 21E_2(q)E_4(q) + 4E_6(q))}{362880\prod_{n\ge1}(1-q^n)^{12}} \ ,$$

$$
\begin{aligned}
Z_{0,2}(q) = & \frac{E_2(q)E_4(q)^2 + 2E_4(q)E_6(q)}{\prod_{n\geq 1}(1-q^n)^{24}} \;, \\
Z_{1,2}(q) = & \frac{10E_2(q)^2E_4(q)^2 + 9E_4(q)^3 + 24E_2(q)E_4(q)E_6(q) + 5E_6(q)^2}{1152\prod_{n\geq 1}(1-q^n)^{24}} \;,
\end{aligned}
$$

$$
\begin{aligned}
Z_{2,2}(q) = & \big(190E_2(q)^3E_4(q)^2 + 417E_2(q)E_4(q)^3 + 540E_2(q)^2E_4(q)E_6(q) + \\
& 356E_4(q)^2E_6(q) + 225E_2(q)E_6(q)^2\big)\frac{1}{207360\prod_{n\geq 1}(1-q^n)^{24}} \;, \\
Z_{3,2}(q) = & \big(2275E_2(q)^4E_4(q)^2 + 8925E_2(q)^2E_4(q)^3 + 3540E_4(q)^4 + \\
& 7560E_2(q)^3E_4(q)E_6(q) + 14984E_2(q)E_4(q)^2E_6(q) + \\
& 4725E_2(q)^2E_6(q)^2 + 4071E_4(q)E_6(q)^2\big)\frac{1}{34836480\prod_{n\geq 1}(1-q^n)^{24}} \;.
\end{aligned}
$$

For $n = 1$ a closed formula valid for all genus is known in [HST1].

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2 Generating function and affine E_8 Weyl orbits

2.1 Notations. Let S be a rational elliptic surface, i.e. $\mathbf{P}^2$ blown up at nine base points of two generic cubics. We denote by e_i the cohomology class of exceptional curve D_i $(i = 1, \cdots, 9)$. Let H be the pullback of the class of a line in $\mathbf{P}^2$. The second cohomology $H^2(S, \mathbf{Z})$ is generated by $H, e_1, \cdots, e_9$;

$$H^2(S, \mathbf{Z}) = \mathbf{Z}H \oplus \mathbf{Z}e_1 \oplus \cdots \oplus \mathbf{Z}e_9 \;.$$

Due to Poincaré duality, $H^2(S, \mathbf{Z})$ becomes unimodular lattice with respect to the natural intersection pairing (cup product) $(*, **) : H^2(S, \mathbf{Z}) \times H^2(S, \mathbf{Z}) \to \mathbf{Z}$. S has an elliptic fibration $f : S \to \mathbf{P}^1$ with the class of the fiber given by

$$F = 3H - e_1 - e_2 - \cdots - e_9 \;.$$

In this note we fix an exceptional curve D_9 as the zero section. Then it is known that the orthogonal lattice,

$$\langle e_9, F\rangle^{\perp} := \{ x \in H^2(S, \mathbf{Z}) \mid (x, e_9) = (x, F) = 0 \,\}$$

is isomorphic to the lattice $E_8(-1)$, i.e. the E_8 root lattice with its pairing multiplied by -1.

2.2 Root system. Let V be a real vector space and V^* be its dual. A finite set B of linearly independent vectors in V together with an injection $\vee : B \to V^*, \alpha \to \alpha^\vee$ is called *root basis* if the following conditions are satisfied: (i) $B^\vee = \{\alpha^\vee | \alpha \in B\}$ are linearly independent, (ii) $\alpha^\vee(\alpha) = -2$ for all α, (iii) $\beta^\vee(\alpha), \alpha \neq \beta$, are nonnegative integers, (iv) $\beta^\vee(\alpha) = 0$ implies $\alpha^\vee(\beta) = 0$. A root basis is called *symmetric* if $\alpha^\vee(\beta) = \beta^\vee(\alpha)$ holds.

When V is equipped with a non-degenerate pairing $(\ , \ ) : V \times V \to \mathbf{R}$ and we define $\vee : B \to V^*$ by

$$\alpha^\vee(x) = (\alpha, x) \ \ , \ \ (x \in V), \tag{2.1}$$

then the first property (i) is easily verified, and also $\alpha^\vee(\beta) = \beta^\vee(\alpha)$. We will soon restrict our attention to a root basis B in $V = H^2(S, \mathbf{Z}) \otimes \mathbf{R}$ with the injection $\vee$ defined by the nondegenerate cup product.

Let (B, V) be a symmetric root basis and write $B = \{\alpha_0, \alpha_1, \cdots, \alpha_r\}$. The (symmetric) matrix

$$A := (a_{ij}) = (\alpha_i^\vee(\alpha_j))_{0 \le i,j \le r} \tag{2.2}$$

is called the *Cartan matrix* of B. We may define a lattice structure on the group $Q = \mathbf{Z}\alpha_0 + \mathbf{Z}\alpha_1 + \cdots \mathbf{Z}\alpha_r$ by setting the bilinear form $(\alpha_i, \alpha_j)_Q = a_{ij}$. This is called the *root lattice* of B. Note that when $\vee$ is defined by (2.1), the bilinear form $(\, , \,)_Q$ on the root lattice coincides with the pairing $(\ , \ )$ on V restricted to $Q \subset V$. (However, it should be noted that the restriction of the nondegenerate pairing to Q is not necessarily nondegenerate on Q.) For any $\alpha_i \in B$, we define a *fundamental reflection* by

$$s_i(x) := s_{\alpha_i}(x) = x + \alpha_i^\vee(x)\alpha_i \quad (x \in V). \tag{2.3}$$

Since $a_{ij} = \alpha_i^\vee(\alpha_j) = a_{ji}$, one may verify that s_i is an element of the orthogonal group $O(Q)$ of the root lattice Q. The *Weyl group* of B is a discrete subgroup of $O(Q)$ which is generated by fundamental reflections. The *fundamental Weyl chamber* C is defined by

$$C = \{x \in V | \alpha^\vee(x) > 0 \ (\alpha \in B)\},$$

and $w(C)$ for some $w \in W$ is called simply a *chamber*. For each subset $Z \subset B$, we define a *fundamental facet* by

$$Facet_Z := \{x \in V | \alpha^\vee(x) = 0 \text{ for } \alpha \in Z \text{ and } \ \alpha^\vee(x) > 0 \text{ for } \alpha \in B \setminus Z\}.$$

Note that $Facet_\phi = C$ and the closure $\bar{C}$ of C is the disjoint union of the fundamental facets. The W-orbit of $\bar{C}$ is called the *Tits cone* and denoted $I := \bigcup_{w \in W} w(\bar{C})$. Tits cone is a convex cone in V. It is known that the Weyl group acts properly discontinuously on the interior $\mathring{I}$ of I and $\bar{C}$ is a fundamental domain for this action. Also it is known that the Weyl group acts simply and transitively on the set of chambers, $\{w(C) | w \in W\}$.

The elements Λ_j in V satisfying $\alpha_i^\vee(\Lambda_j) = \delta_{ij}$ are called *fundamental weights*. Note that fundamental weights are determined up to an elements F_B.

2.3 Root system defined in $H^2(S, \mathbf{Z})$. Here we introduce a root basis in $V = H^2(S, \mathbf{Z}) \otimes \mathbf{R}$ following [Lo]. Let us define $\alpha_0 = e_8 - e_9, \alpha_i = e_i - e_{i+1}$ $(1 \le i \le 7)$ and $\alpha_8 = H - e_1 - e_2 - e_3$ and consider a finite set in V

$$B = \{\alpha_0, \alpha_1, \cdots, \alpha_8\} \ \ .$$

Since the cup product on $H^2(S,\mathbf{Z})$ is nondegenerate, so is its scalar extension to V. By this nondegenerate form and (2.1), we define the injective map $\vee : B \to V^*$. Then it is easy to verify that (B,V) is in fact a root basis defined in 2.2, and also that the Cartan matrix of B defined by (2.2) coincides with that of the affine $\hat{E}_8(-1)$[Kac]. (In fact, the root basis is of *affine type*, which characterized by the properties: (i) it is irreducible, (ii) the Cartan matrix is of corank one and (iii) $W_X := \langle s_\alpha | \alpha \in X\rangle$ is finite group for any proper subset $X \subset B$. See [Kac] for more details.) The Weyl group associated to this root basis is called *affine Weyl group* of $E_8(-1)$, and will be denoted by $W_{\hat{E}_8}$. By definition, the root lattice $(Q,(\ ,\ )_Q)$ is naturally a sublattice of $(H^2(S,\mathbf{Z}),(\ ,\ ))$, and we may verify directly that

$$H^2(S,\mathbf{Z}) = Q \oplus \mathbf{Z}F = \mathbf{Z}\alpha_0 \oplus \mathbf{Z}\alpha_1 \oplus \cdots \mathbf{Z}\alpha_8 \oplus \mathbf{Z}F,$$

as a lattice. Also we verify $Facet_B = \mathbf{R}\,F$. The affine Weyl group is an subgroup of $O(Q)$, and also may be regarded as a subgroup of $O(H^2(S,\mathbf{Z}))$ since it acts trivially on $\mathbf{Z}\,F$.

The Tits cone I is known in [Lo, Proposition (3.9)] to be the union of the half space $\{x \in V | (x,F) > 0\}$ and the facet $Facet_B = \mathbf{R}\,F$.

The fundamental weights $\Lambda_i \in V$ (, s.t. $\alpha_i^\vee(\Lambda_j) = \delta_{ij}$,) are determined up to $Facet_B$. Since the lattice $H^2(S,\mathbf{Z})$ is unimodular, we may take Λ_i in $H^2(S,\mathbf{Z})$ up to $\mathbf{Z}\,F$. Fixing this ambiguity by hand, we define

$$\begin{gathered}\Lambda_0 = e_9,\ \ \Lambda_1 = H - e_1,\ \ \Lambda_2 = 2H - e_1 - e_2,\ \ \Lambda_3 = 3H - e_1 - e_2 - e_3, \\ \Lambda_4 = 3H - e_1 - e_2 - e_3 - e_4\ \ ,\ \ \Lambda_5 = F + e_6 + e_7 + e_8 + e_9\ \ , \\ \Lambda_6 = F + e_7 + e_8 + e_9\ \ ,\ \ \Lambda_7 = F + e_8 + e_9\ \ ,\ \ \Lambda_8 = H\ . \end{gathered} \tag{2.4}$$

Remark. (1)We note that the zero-th root may be written by $\alpha_0 = F - \theta$, where $\theta = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$ is the *highest root* of the (classical) root basis $B^{cl} := \{\alpha_1, \cdots, \alpha_8\}$.

(2) We may extend linearly the injective map on a (symmetric) root basis $\vee : B \to V^*$ to the root lattice $\vee : Q \to V^*$, $\sum_k m_k\alpha_k \mapsto \sum_k m_k \alpha_k^\vee$. Then the simple reflection s_{α_i} defined for $\alpha_i \in B$ by (2.3) may be extended to r_α for $\alpha \in Q$ with $\alpha^\vee(\alpha) = -2$. The highest root θ is a so-called *real root*, i.e. a root α such that $\alpha = w(\alpha_i)$ for some $w \in W$ and $\alpha_i \in B$. From this, we see $\theta^\vee(\theta) = (\theta,\theta) = -2$ and also $r_\theta \in W$ since we have the relation $r_{\alpha_i} \circ r_{\alpha_j} \circ r_{\alpha_i} = r_{r_{\alpha_i}(\alpha_j)}$. Now we define *translation* $t_\gamma : Q \to Q$ $(\gamma \in E_8(-1))$ by

$$t_\gamma(\beta) = \beta + (F,\beta)\gamma - \{\frac{1}{2}(F,\beta)(\gamma,\gamma) + (\beta,\gamma)\}F\ \ , \tag{2.5}$$

which satisfy $t_\gamma \circ t_{\gamma'} = t_{\gamma+\gamma'}$, and consider a group of translations $T := \{t_\gamma |\ \gamma \in E_8(-1)\}$. Then we may verify the following relations;

$$r_{\alpha_0} \circ r_\theta = t_{-\theta}\ \ ,\ \ r_\alpha \circ t_{-\theta} \circ r_\alpha = t_{-r_\alpha(\theta)}.$$

In fact, it is known (see e.g. [Kac]) that the affine Weyl group $W_{\hat{E}_8}$ is a semi-direct product of the translation group T and the *classical* Weyl group $W_{E_8} := \langle r_{\alpha_1}, \cdots, r_{\alpha_8}\rangle$;

$$W_{\hat{E}_8} = W_{E_8} \ltimes T\ \ . \tag{2.6}$$

2.4 $Z_{g;n}$ and orbit decompositions. Let $s_{\alpha_i}(0 \le i \le 8)$ be reflections defined in (2.3), and consider their actions on the cohomology basis $s_{\alpha_i}: H, e_1, \cdots, e_9 \mapsto s_{\alpha_i}(H), s_{\alpha_i}(e_1), \cdots, s_{\alpha_i}(e_9)$. For $i = 0, \cdots, 7$, the actions are simply interchanges $e_j \leftrightarrow e_{j+1}$. For s_{α_8}, we have

$$s_{\alpha_8}(H) = 2H - e_1 - e_2 - e_3 \ , \ s_{\alpha_8}(e_1) = H - e_2 - e_3 \ ,$$
$$s_{\alpha_8}(e_2) = H - e_1 - e_3 \ , \ s_{\alpha_8}(e_3) = H - e_1 - e_2 \ ,$$

and $s_{\alpha_8}(e_k) = e_k$ $(4 \le k \le 9)$. Here we see, for example, that the class $s_{\alpha_8}(e_1)$ represents that of the line passing through the points p_2 and p_3 where we blow up in $\mathbf{P}^2$ to obtain S. Each class represents a smooth rational curve with self-intersection -1, which can be contracted. Therefore for each s_{α_i} $(0 \le i \le 8)$, the classes $s_{\alpha_i}(e_1), \cdots, s_{\alpha_i}(e_9)$ represent the -1 curves which we can contract. Contracting these 9 curves to points $p'_1, \cdots, p'_9$, we obtain $'\mathbf{P}^2$ which is birational to the original $\mathbf{P}^2$ used to define S. From this viewpoint, we may regard the class $s_{\alpha_i}(H) =: H'$ as the pullback of the class of a line in $'\mathbf{P}^2$, and $s_{\alpha_i}(e_k) =: e'_k$ as the class of the exceptional divisor for the blowing up at p'_k. The configuration of $p'_1, \cdots, p'_9$ in $'\mathbf{P}^2$ differs from that of $p_1, \cdots, p_9$ in $\mathbf{P}^2$, and thus blowing up these points results in rational elliptic surface S' with different complex structure from S. However by construction, S' is identical to S. That is, there is an isomorphism between the two rational elliptic surfaces S and S' with different complex structures. (See [Lo, Theorem (5.3)] for Torelli type theorem for rational surfaces.)

Now we may combine this isomorphism with the invariance of Gromov-Witten invariants under the deformations. To describe it precisely, let us write $N_g^S(\beta)$ $(\beta \in H^2(S, \mathbf{Z}))$ the Gromov-Witten invariants for the surface S, and similarly $N_g^{S'}(\beta')$ $(\beta' \in H^2(S', \mathbf{Z}))$ for the surface S'. For example, let us assume S' is defined as above for the reflection s_{α_8} and use the notations for $H, e_1, \cdots, e_9$ and $H', e'_1, \cdots, e'_9$ introduced above. Then we have, for example;

$$N_g^S(e_1) = N_g^{S'}(e'_1) = N_g^S(H - e_2 - e_3) \ ,$$

where the first equality is the invariance under the deformations and the second follows the isomorphism $\Phi : S \cong S'$. In the exactly same way, we have the equality $N_g^S(\beta) = N_g^S(s_{\alpha_i}(\beta))$ for all reflections s_{α_i} $(i = 0, 1, \cdots, 8)$. Since the affine Weyl group $W_{\hat{E}_8}$ is generated by the reflections s_{α_i}, we have:

Proposition 2.1.

$$N_g(\beta) = N_g(\omega(\beta)) \ , \ (\beta \in H^2(S, \mathbf{Z}), \omega \in W_{\hat{E}_8})$$

In what follows we will utilize this invariance to study the (solutions of the) holomorphic anomaly equation (1.4). As a result, in the next section, we will determine the numbers $N_g(\beta)$ for several $\beta \in H^2(S, \mathbf{Z})$. The idea is simply to make the orbit decomposition of the generating function:

Definition 2.2. We define the *character of the generating function (or simply generating function)*, $\mathcal{Z}_{g;n} : H^2(S, \mathbf{Z}) \otimes \mathbf{C} \to \mathbf{C}^*$ by

$$\mathcal{Z}_{g;n} := \sum_{\beta \in H^2(S,\mathbf{Z}), (\beta, F) = n} N_g(\beta) e^{2\pi\sqrt{-1}\beta} \ \ (n > 0) \tag{2.7}$$

where $e^{2\pi\sqrt{-1}\beta}$ is the character defined by $e^{2\pi\sqrt{-1}\beta}(c) := e^{2\pi\sqrt{-1}(\beta, c)}$ for $c \in H^2(S, \mathbf{Z}) \otimes \mathbf{C}$ with the cup product (,) extended to over $\mathbf{C}$.

Remark (1) The condition $(\beta, F) = n$ restricts the classes β to those of n-sections. Since this condition is obviously invariant under the Weyl group action, we define $\mathcal{Z}_{g;n}$ restricting the sum over β's of n-sections.

(2) The generating function $Z_{g;n}(q)$ $(q = e^{2\pi\sqrt{-1}\tau})$ introduced in (1.2) is the character $\mathcal{Z}_{g;n}$ evaluated by $\tau\sigma$ with a class of (positive) section $\sigma = e_9 + F$, i.e., $Z_{g;n}(q) = \mathcal{Z}_{g;n}(\tau\sigma)$.

By the general theory of Gromov-Witten invariants[KM], to have non-vanishing Gromov-Witten invariants $N_g(\beta)$ it is necessary that β represents a class of effective and connected (but not necessarily irreducible) divisor. For connected and effective divisor class β, we have $(\beta, F) \geq 0$ and the equality holds only if $\beta = kF$ for some positive integer k. If we omit these rather trivial cases $\beta = kF$ from our consideration, we see that the condition $(\beta, F) > 0$ coincides with that β belongs to an integral class contained in the Tits cone. Now it is obvious from Proposition 2.1 that the invariant $N_g(\beta)$ is determined by its value for β in the closure $\bar{C}$ of the fundamental Weyl chamber.

The integral elements λ in the fundamental Weyl chamber are called *dominant weight of level* $n(>0)$ if they satisfies $(\lambda, F) = n$. If λ is dominant integral weight of level n, then so is $\lambda + aF$ for arbitrary integer a. To choose this a as small as possible, we impose the following numerical conditions;

(1) the arithmetic genus

$$g_{\lambda'} = \tfrac{1}{2}\{(\lambda', \lambda') + 2 - (\lambda', F)\} \geq 0 \ ,$$

and $g_{\lambda'}$ is minimum.

(2) if $n \geq 2$ then $d \geq 1$ and $a_1, \cdots, a_9 \geq 0$ for $\lambda' = dH - a_1e_1 - \cdots - a_9e_9$.

We will call the dominant weights satisfying (1) and (2) *minimal*.

Definition 2.3. We denote the set of minimal dominant weights of level n by $\mathcal{P}^{min}_{+,n}$, i.e. $\mathcal{P}^{min}_{+,n} := \{\lambda \in H^2(S, \mathbf{Z}) \mid (\lambda, \alpha_i) \geq 0 \ (i = 1, \cdots, 8), (\lambda, F) = n, \lambda\text{: minimal}\}$.

It is easy to verify that each fundamental weight Λ_i introduced in (2.4) is minimal as well as dominant. Note that addition of minimal dominant weights results in a dominant weight, however the minimality of weights is not preserved. Now it will be convenient to define the addition among the minimal dominant weight by

$$\lambda + \lambda' := \text{ minimal dominant weight in } \lambda + \lambda' + \mathbf{Z}F, \tag{2.8}$$

for minimal dominant weights λ, λ'. Hereafter we write the fundamental weights $\Lambda_0, \Lambda_1 \cdots, \Lambda_8$ by $\lambda_0, \lambda_1, \cdots, \lambda_8$ with this understanding for the addition. In Table 1, elements in $\mathcal{P}^{min}_{+,n}$ are listed for $n \leq 4$.

Now we are ready to accomplish the orbit decomposition of the character (2.7):

Proposition 2.4. *The character $\mathcal{Z}_{g;n}$ is decomposed into the orbits by*

$$\mathcal{Z}_{g;n} = \sum_{\lambda \in \mathcal{P}^{min}_{+,n}} \mathcal{Z}_{g,\lambda} \, P_\lambda \ , \tag{2.9}$$

where

$$\mathcal{Z}_{g,\lambda} := \sum_{a \in \mathbf{Z}} N_g(\lambda + aF) e^{2\pi\sqrt{-1}(\lambda + aF)} \ , \ P_\lambda := \sum_{\omega \in W_{\hat{E}_8}(\lambda)} e^{2\pi\sqrt{-1}(\omega(\lambda) - \lambda)} \ , \tag{2.10}$$

with $W_{\hat{E}_8}(\lambda) := W_{\hat{E}_8}/(\text{stabilizer of } \lambda)$.

Proof Since the integral classes β with $(\beta, F) = n > 0$ are contained in the Tits cone, for each Weyl orbit we may take a unique representative in the closure $\bar{C}$ of the fundamental Weyl chamber. Then we have

$$\begin{aligned}\mathcal{Z}_{g;n} &= \sum_{\beta \in H^2(S,\mathbf{Z}),(\beta,F)=n} N_g(\beta) e^{2\pi\sqrt{-1}\beta} \\ &= \sum_{\lambda \in \mathcal{P}^{\min}_{+,n}} \sum_{a \in \mathbf{Z}} \sum_{\omega \in W_{\hat{E}_8}(\lambda)} N_g(\omega(\lambda) + aF) e^{2\pi\sqrt{-1}(\omega(\lambda)+aF)} \\ &= \sum_{\lambda \in \mathcal{P}^{min}_{+,n}} \left(\sum_{a \in \mathbf{Z}} N_g(\lambda + aF) e^{2\pi\sqrt{-1}(\lambda+aF)} \right) \left(\sum_{w \in W_{E_8}(\lambda)} e^{2\pi\sqrt{-1}(\omega(\bar{\lambda})-\lambda)} \right),\end{aligned}$$

where we remark that if λ sits in the walls of $\bar{C}$, it has nontrivial stabilizers. Also the summation over a has in fact lower bound(, see Remark below). □

Remark By general property of Gromov-Witten invariants, we have $N_g(\lambda + aF) = 0$ unless $\lambda + a\,F$ is effective. Since $\lambda + a\,F$ is not effective for $a << 0$, we have a lower bound a_0 for the summation over $a \in \mathbf{Z}$ in the above proposition. For the examples, which are listed in this paper (Table 2–5), the lower bounds turn out in fact to be zero, i.e. $a_0 = 0$.

The character P_λ ($\lambda \in \mathcal{P}^{min}_{+,n}$) represents a summation over the Weyl orbit which is parametrized by $\lambda + aF$ ($a \geq 0$). We call the character P_λ, which is independent of a, *multiplicity* of the invariants $N_g(\lambda + aF) = N_g(\omega(\lambda + aF))$.

n=1	(0;0,0,0,0,0,0,0,0,-1)=λ_0	g=0
n=2	(1;1,0,0,0,0,0,0,0,0)=λ_1	g=0
	(3;1,1,1,1,1,1,1,0,0)=λ_7	g=1
	(6;2,2,2,2,2,2,2,2,0)=$2\lambda_0$	g=2
n=3	(1;0,0,0,0,0,0,0,0,0)=λ_8	g=0
	(3;1,1,1,1,1,1,0,0,0)=λ_6	g=1
	(4;2,1,1,1,1,1,1,1,0)=$\lambda_0+\lambda_1$	g=2
	(6;2,2,2,2,2,2,2,1,0)=$\lambda_0+\lambda_7$	g=3
	(9;3,3,3,3,3,3,3,3,0)=$3\lambda_0$	g=4

n=4	(2;1,1,0,0,0,0,0,0,0)=λ_2	g=0
	(3;1,1,1,1,1,0,0,0,0)=λ_5	g=1
	(4;2,1,1,1,1,1,1,0,0)=$\lambda_1+\lambda_7$	g=2
	(4;1,1,1,1,1,1,1,1,0)=$\lambda_0+\lambda_8$	g=3
	(5;3,1,1,1,1,1,1,1,1)=$2\lambda_1$	g=3
	(6;2,2,2,2,2,2,2,0,0)=$2\lambda_7$	g=3
	(6;2,2,2,2,2,2,1,1,0)=$\lambda_0+\lambda_6$	g=4
	(7;3,2,2,2,2,2,2,2,0)=$2\lambda_1$	g=5
	(9;3,3,3,3,3,3,3,2,0)=$2\lambda_7$	g=6
	(12;4,4,4,4,4,4,4,4,0)=$4\lambda_0$	g=7

Table 1. Minimal dominant weights in $\mathcal{P}^{min}_{+,n}$ up to level $n = 4$. $(d; a_1, a_2, \cdots, a_9)$ represents the minimal domionant weight $\lambda = dH - a_1e_1 - \cdots - a_9e_9$. We also list the arithmetic genus $g = 12((\lambda,\lambda) + 2 - (\lambda, F)) = (d-1)(d-2)2 - \sum_{i=1}^{9} a_i(a_i-1)2$.

2.5 The multiplicity functions P_λ**.** The multiplicity P_λ determines corresponding multiplicity function $P_\lambda(\tau, u_1, \cdots, u_8)$ when we evaluate it by $u_1\alpha_1 + \cdots + u_8\alpha_9 + \tau(e_9 + F) \in H^2(S, \mathbf{Z})$, i.e.,

$$P_\lambda(\tau, u_1, \cdots, u_8) := P_\lambda(u_1\alpha_1 + \cdots + u_8\alpha_9 + \tau(e_9 + F)).$$

As we observe in (2.10), there is a similarity between P_λ and the numerater of the Weyl-Kac character formula for the integrable representation of affine Kac-Moody algebra[Kac]. As in the case for the Weyl-Kac character formula, we may write the multiplicity functions, at least formally, in terms of the theta function of the E_8 lattice.

Proposition 2.5. *For $n\lambda_0 \in \mathcal{P}^{min}_{+,n}$, we have*

$$P_{n\lambda_0}(\tau, u_1, \cdots, u_8) = \Theta_{E_8}(n\tau, nu_1, \cdots, nu_8) \ ,$$

where $\Theta_{E_8}(\tau, u_1, \cdots, u_8) = \sum_{l \in E_8} e^{2\pi\sqrt{-1}\left(\frac{(l,l)}{2}\tau + (l, u_1\alpha_1 + \cdots + u_8\alpha_8)\right)}$ is the theta function of the E_8-lattice.

Proof The affine Weyl group $W_{\hat{E}_8}$ is represented by a semi-direct product of the translation group $T = \{t_\gamma | \gamma \in E_8(-1)\}$ and the classical Weyl group generated by $s_{\alpha_1}, \cdots, s_{\alpha_8}$. Since the classical Weyl group is exactly the stabilizer of $n\lambda_0 \in \mathcal{P}^{min}_{+,n}$, we have from (2.10),

$$\begin{aligned} P_{n\lambda_0} &= \sum_{\omega \in W_{\hat{E}_8}(n\lambda_0)} e^{2\pi\sqrt{-1}(\omega(n\lambda_0) - n\lambda_0)} = \sum_{\gamma \in E_8(-1)} e^{2\pi\sqrt{-1}(t_\gamma(n\lambda_0) - n\lambda_0)} \\ &= \sum_{\gamma \in E_8(-1)} e^{2\pi\sqrt{-1}\left(n\gamma - \frac{(\gamma,\gamma)}{2} nF\right)} \ . \end{aligned}$$

Evaluating the character with $u_1\alpha_1 + \cdots + u_8\alpha_8 + \tau(e_9 + F)$, we obtain the desired result. □

Explicit form of the function $P_\lambda(\tau, u_1, \cdots, u_2)$ for general λ contains summation over an non-trivial group $W_{\hat{E}_8}(\lambda)$ and complicated in general. However for lower levels n and special values for $u_1, \cdots, u_8$, we may have simple form for the multiplicity function. For example, in case of $n = 2$, we have three elements $2\lambda_0, \lambda_1, \lambda_7$ in $\mathcal{P}^{min}_{+,2}$, and the multiplicity functions

$$P_0(\tau) := P_{2\lambda_0}(\tau(e_9 + F)) \ , \ P_{even}(\tau) := P_{\lambda_1}(\tau(e_9 + F)) \ , \ P_{odd}(\tau) := P_{\lambda_7}(\tau(e_9 + F)),$$

have the following simple forms, which were first appeared in [MNVW][Yo].

Proposition 2.6. *([MNVW],[Yo]) For the multiplicity functions defined above, we have;*

$$\begin{aligned} P_{even}(\tau) &= \left(\frac{E_4(\tau) + E_4(\tau + \frac{1}{2})}{2} - E_4(2\tau) \right) q^{-1} \ , \\ P_{odd}(\tau) &= \left(\frac{E_4(\tau) - E_4(\tau + \frac{1}{2})}{2} \right) q^{-\frac{1}{2}} \ , \ \ P_0(\tau) = E_4(2\tau) \ , \end{aligned}$$

where $E_4(\tau)$ is the Eisenstein series of weight four which is a special value of Θ_{E_8}, i.e. $E_4(\tau) = \Theta_{E_8}(\tau, 0, \cdots, 0)$.

Since derivation of these forms, and further generalizations to $n = 3$, from our definition (2.10) are easy, we do not reproduce them here.

2.6 Theta function Θ_{E_8}. Here we summarize a convenient realization of the theta function $\Theta_{E_8}(\tau, u_1, \cdots, u_8)$, which are often used in the literatures. To do this, let us consider $\mathbf{R}^9$ with its orthonormal basis $\varepsilon_1, \cdots, \varepsilon_9$, $(,(\varepsilon_i, \varepsilon_j) = \delta_{ij})$. In this space we realize the E_8 lattice $\sum_{i=1}^{9} \mathbf{Z}\alpha_i$ by setting $\alpha_1 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_7 + \varepsilon_8)$, $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ $(2 \le i \le 7)$, $\alpha_8 = \varepsilon_1 + \varepsilon_2$. Then the E_8 theta function may be evaluated to

$$\Theta_{E_8}(\tau, u_1, \cdots, u_8) = \sum_{\gamma \in E_8} e^{2\pi\sqrt{-1}\left(\frac{(\gamma,\gamma)}{2} + (\gamma, u_1\alpha + \cdots + u_8\alpha_8)\right)} = \frac{1}{2}\sum_{i=1}^{4}\prod_{j=1}^{8}\theta_l(\tau, z_j) \ , \tag{2.11}$$

with

$$\theta_1(\tau, z) := i\sum_{n\in\mathbf{Z}}(-1)^n q^{(n+12)^2} y^{n+12} \ , \qquad \theta_3(\tau, z) := \sum_{n\in\mathbf{Z}} q^{n^2} y^n \ ,$$
$$\theta_2(\tau, z) := \sum_{n\in\mathbf{Z}} q^{(n+12)^2} y^{n+12} \ , \qquad \theta_4(\tau, z) := \sum_{n\in\mathbf{Z}}(-1)^n q^{n^2} y^n \ ,$$

where $q = e^{2\pi\sqrt{-1}\tau}, y = 2\pi\sqrt{-1}z$ and $z_1, \cdots, z_8$ are determined by the relation $\sum_{i=1}^{8} u_i\alpha_i = \sum_{j=1}^{8} z_j\varepsilon_j$. Hereafter we denote the right hand side of (2.11) by $\Theta^{\mathbf{Z}}_{E_8}(\tau, z_1, \cdots, z_8)$. Namely, $\Theta^{\mathbf{Z}}_{E_8}(\tau, z_1, \cdots, z_8)$ and $\Theta_{E_8}(\tau, u_1, \cdots, u_8)$ should be related by the linear relation $\sum_{i=1}^{8} u_i\alpha_i = \sum_{j=1}^{8} z_j\varepsilon_j$.

3 Orbit decomposition and BPS numbers

In this section we study the solutions of the holomorphic anomaly equation (1.4). So far we do not have general proof about that our holomorphic anomaly equation (1.4) really evaluates the generating function of the Gromov-Witten invariants defined by (1.1), although we may verify that it produces consistent predictions $N_g(\beta)$ for many β. Under this circumstance, our approach is *to assume* that the generating function defined in (2.7), or more precisely $\mathcal{Z}_{g;n}(\tau(e_9 + F))$, is a solution of the holomorphic anomaly equation (1.4).

3.1 Vanishing conditions. As we see in (1.4), in order to solve the holomorphic anomaly equation we need to fix "integration constants $f_{2g+6n-2}(E_4, E_6)$", the polynomial ambiguity which appears in the integration. This polynomial ambiguity is sometimes called *holomorphic ambiguity* in literatures. We see that the following requirements for Gromov-Witten invariants (BPS numbers) provide conditions to fix this ambiguity. The meaning of BPS numbers will be summarized briefly in section 3.2.

Definition 3.1. (Vanishing conditions on BPS numbers) We define the BPS number $n_h(\beta)$, for β satisfying $(\beta, F) \geq 1$, by the relation (1.5). Then we impose $n_h(\beta) = 0$ unless the following conditions are satisfied:
(i) $d \geq 1, a_1, \cdots, a_9 \geq 0$ for $\beta = dH - a_1e_1 - \cdots - a_9e_9$ if $(\beta, F) \geq 2$,
(ii) $\beta = e_i$ $(i = 1, \cdots, 9)$ or $d \geq 1, a_1, \cdots, a_9 \geq 0$ if $(\beta, F) = 1$,
(iii) $0 \leq h \leq \frac{1}{2}\{(\beta, \beta) - (\beta, F) + 2\}$.

In order to impose the vanishing conditions on $Z_{g;n}$, it is useful to introduce the following notations (with $q = e^{2\pi\sqrt{-1}\tau}$):

$$\tilde{Z}_{h;n}(q) := \sum_{\beta\in H^2(S,\mathbf{Z}), (\beta,F)=n} n_h(\beta) q^{(\beta, e_9+F)} \ ,$$

which is related to $Z_{g;n}(q)$ by

$$Z_{g;n}(q) = \sum_{k|n} k^{2g-3}\sum_{h=0}^{g} C(h, g-h)\tilde{Z}_{h;n/k}(q^k) \ .$$

Since from the defining relation (1.5), we have $n_h(\beta)=n_h(w(\beta))$ ($w\in W_{\hat{E}_8}$) and therefore we may consider the orbit decomposition $\tilde{Z}_{g;n}(q)=\sum_{\lambda\in\mathcal{P}^{min}_{+,n}}\tilde{Z}_{h;\lambda}(q)P_\lambda(q)$ in a similar way to $Z_{g;\lambda}(q)$. In this case, the function $\tilde{Z}_{h,\lambda}(q)$ have the following form,

$$\tilde{Z}_{h,\lambda}(q)=\sum_{a\geq a_0} n_h(\lambda+a\,F)q^{(\lambda+a\,F,e_9+F)} \quad . \tag{3.1}$$

Here we note that, from the vanishing conditions (i),(ii) and the definition of the minimal dominant weights $\lambda\in\mathcal{P}^{min}_{+,n}$, the sum over $a\geq a_0$ is in fact restricted to $a\geq a_0\geq 0$. Then since $(\lambda+aF,e_9+F)=(\lambda,e_9)+n+a\geq n$ for $\lambda\neq e_9$, we see that $\tilde{Z}_{h,\lambda}(q)$ starts from an order higher than q^n for $\lambda\neq e_9$. (The case $\lambda=e_9\in\mathcal{P}^{min}_{+,n}$ is possible only for $n=1$. For simplicity, we omit this case from our consideration in what follows.) Now since $P_\lambda(q)=1+$(higher order terms in q) by (2.10), we see that

($*$) *For $n\geq 2$ the q-expansion of $\tilde{Z}_{h,n}(q)$ starts from an order higher than q^n.*
This is an easy way to impose the vanishing condition (i),(ii), and is equivalent to the *gap condition* imposed for $Z_{g=0,n}$ in [MNVW]. The third condition (iii) further restricts the lower bound a_0 in (3.1) depending h, and as a result, we have much refined conditions for the q-expansion of $\tilde{Z}_{h,n}(q)$. Since the arguments are straightforward, we omit its details here.

The vanishing condition (*) and its refinement with the condition (iii) are those what we have in order to fix the "integration constants" $f_{2g+6n-2}(E_4,E_6)$. In the case of $g=0$, the conditions from the vanishing condition (*) grow linearly in n whereas the dimensions of the integration constants $f_{6n-2}(E_4,E_6)$, i.e. dimensions of modular forms of weight $6n-2$, do not. Therefore the existence of the solution satisfying the vanishing condition is highly non-trivial. In ref.[MNW], the existence was shown by constructing the solutions explicitly for $g=0$. This situation is similar for our higher genus generalization (1.4). However the corresponding explicit closed formula of the solutions has been obtained only for $g=1$. For $g\geq 2$, the existence of the solution satisfying the vanishing conditions are verified for lower values of g and n, e.g. $g,n\leq 10$. Some of them are displayed in the end of the section 1.

3.2 Orbit decomposition $\mathbf{n\leq 2}$. The case for $n=1$, the orbit decomposition is rather trivial since the set $\mathcal{P}^{min}_{+,1}$ consists only one element λ_0. Then, for example, the initial data $Z_{0,1}(\tau)=\frac{q^{\frac12}E_4(q)}{\eta(q)^{12}}$ in (1.4) is decomposed to

$$Z_{0,1}(\tau)=\frac{q^{\frac12}}{\eta(q)^{12}}P_{\lambda_0}(\tau,0,\cdots,0) \quad ,$$

where, by Proposition 2.5, $P_{\lambda_0}(\tau,0,\cdots,0)=\Theta_{E_8}(\tau,0,\cdots,0)=E_4(q)$. This implies that

$$\mathcal{Z}_{0,\lambda_0}(\tau(e_9+F))=\sum_{a\geq 0}N_0(\lambda_0+a\,F)q^{(\lambda_0+a\,F,(e_9+F))}=1\prod_{m>0}(1-q^m)^{12} \quad ,$$

which is in the same form, except the power 12 replaced by 24, as the counting function for the nodal rational curves in K3 surfaces found in [YZ]. See [HSS],[HST1,2] for detailed interpretations. For higher genus, $Z_{g;1}(\tau)$, the orbit decompositions are simply achieved dividing by the multiplicity function $P_{\lambda_0}(\tau) = E_4(\tau)$, i.e., we simply have $Z_{g,\lambda_0}(\tau) = Z_{g;1}(\tau)(P_{\lambda_0}(\tau))^{-1}$.

For the level $n = 2$ cases, we need to make the following decomposition,

$$Z_{g;n}(\tau) = Z_{g,2\lambda_0}(\tau)P_{2\lambda_0}(\tau) + Z_{g,\lambda_{even}}(\tau)P_{\lambda_{even}}(\tau) + Z_{g,\lambda_{odd}}(\tau)P_{\lambda_{odd}}(\tau) ,$$

where $\lambda_{even} = \lambda_1, \lambda_{odd} = \lambda_7$ (, see Table 1). This decomposition has been done for $g = 0$ in [MNVW][Yo] noticing modular properties of the functions $P_{2\lambda_0}(\tau), P_{\lambda_{even}}(\tau)$ and $P_{\lambda_{odd}}(\tau)$, e.g. $P_{2\lambda_0}(\tau) = E_4(2\tau)$ is a modular form of the group $\Gamma_1(2)$. Since E_2 does not behave modular form, the E_2-dependence of $Z_{g;n}(\tau)$ should be found in $Z_{g,\lambda}(\tau)$. Then using the identity

$$\begin{aligned} P_{\lambda_0}(\tau, u_i)^2 = P_{\lambda_0}(\tau, 0)P_{2\lambda_0}(\tau, u_i) + C_{\lambda_{even}} P_{\lambda_{even}}(\tau, 0)P_{\lambda_{even}}(\tau, u_i) \\ + C_{\lambda_{odd}} P_{\lambda_{odd}}(\tau, 0)P_{\lambda_{odd}}(\tau, u_i), \end{aligned}$$

and linear independence of $P_\lambda(\tau)$'s, we may derive the holomorphic anomaly equation for $Z_{g,\lambda}(\lambda \in \mathcal{P}^{min}_{+,n})$;

$$\frac{\partial Z_{g,\lambda}(\tau)}{\partial E_2} = C_\lambda 24 \sum_{g'+g''=g} Z_{g',\lambda_0}(\tau)Z_{g'',\lambda_0}(\tau)P_\lambda(\tau, 0) + 14 Z_{g-1,\lambda} \ , \qquad (3.2)$$

where $C_\lambda = 1, \frac{q^2}{135}, \frac{q}{120}$, respectively, for $\lambda = 2\lambda_0, \lambda_{even}, \lambda_{odd}$. Integrating (3.2) for $g = 0$, in [MNVW] and [Yo] the following forms are determined;

$$\begin{aligned} &Z_{0,2\lambda_0}(\tau) = 124q\eta(\tau)^{24}\big\{116(4G_2(\tau)^2 - 3G_4(\tau))E_2(\tau) + 18(2G_2(\tau)^2 - 3G_4(\tau))G_2(\tau)\big\} \\ &Z_{0,\lambda_{even}}(\tau) = 124q^2\eta(\tau)^{24}G_4(\tau)\big\{116E_2(\tau) - 18G_2(\tau)\big\} \\ &Z_{0,\lambda_{odd}}(\tau) = 124q^{32}\eta(\tau)^{24}G_4(\tau)^{12}\big\{116G_2(\tau)E_2(\tau) - 132(2G_2(\tau)^2 + 3G_4(\tau))\big\} \ , \end{aligned}$$

where $G_2(\tau) := \theta_3(\tau, 0)^4 + \theta_4(\tau, 0)^4,\ G_4(\tau) := \theta_2(\tau, 0)^8$ are generators of the ring of the modular forms of $\Gamma_1(2)$. Now their argument extends straight forward way to our cases $g \geq 2$. The results are as follows:

Proposition 3.2. *The characters $Z_{g,\lambda}(\tau(F + e_9))$ $(\tilde{\lambda} \in P^{min}_{+,2})$ may be written in terms of the generators $G_2(\tau), G_4(\tau)$ of the modular forms of $\Gamma_1(2)$.*

Here we list the results up to $g = 3$, although calculations continues to higher g as well:

(i) $2\lambda_0$

$$\begin{aligned} Z_{1,2\lambda_0}(\tau) =& 124^2 64q\eta^{24}(20(4G_2^2 - 3G_4)E_2^2 + 48(2G_2^3 - 3G_2G_4)E_2 \\ &+ 28G_2^4 - 27G_2^2G_4 + 27G_4^2) \\ Z_{2,2\lambda_0}(\tau) =& 124^4 20q\eta^{24}\big(380(4G_2^2 - 3G_4)E_2^3 + 1080(2G_2^3 - 3G_2G_4)E_2^2 \\ &+ 3(428G_2^4 - 387G_2^2G_4 + 387G_4^2)E_2 + 356G_2^5 - 636G_2^3G_4 - 432G_2G_4^2\big) \\ Z_{3,2\lambda_0}(\tau) =& 124^5 1120q\eta^{24}\big(36400(4G_2^2 - 3G_4)E_2^4 + 120960(2G_2^3 - 3G_2G_4)E_2^3 \\ &+ 4200(52G_2^4 - 45G_2^2G_4 + 45G_4^2)E_2^2 \\ &+ 64(1873G_2^5 - 3378G_2^3G_4 - 2241G_2G_4^2)E_2 \\ &+ 30444G_2^6 - 54117G_2^4G_4 + 113454G_2^2G_4^2 + 31995G_4^3\big) \end{aligned}$$

(ii) $\lambda_{even} = \lambda_1$

$$
\begin{aligned}
Z_{1,\lambda_{even}}(\tau) &= 124^2 64 q^2 \eta^{24} G_4 \left(20E_2^2 - 48G_2E_2 + 13G_2^2 + 15G_4\right) \\
Z_{2,\lambda_{even}}(\tau) &= 124^4 20 q^2 \eta^{24} G_4 \big(380E_2^3 - 1080G_2E_2^2 + 3(197G_2^2 + 231G_4)E_2 \\
&\qquad - 4(25G_2^3 + 153G_2G_4)\big) \\
Z_{3,\lambda_{even}}(\tau) &= 124^5 1120 q^2 \eta^{24} G_4 \big(36400E_2^4 - 120960G_2E_2^3 + 840(119G_2^2 + 141G_4)E_2^2 \\
&\quad - 128(262G_2^3 + 1611G_2G_4)E_2 + 3(1301G_2^4 + 27726G_2^2G_4 + 11565G_4^2)\big)
\end{aligned}
$$

(iii) $\lambda_{odd} = \lambda_7$

$$
\begin{aligned}
Z_{1,\lambda_{odd}}(\tau) &= 124^2 64 q^{32} \eta^{24} G_4^{12} \left(20G_2E_2^2 - 12(G_2^2 + 3G_4)E_2 + G_2^3 + 27G_2G_4\right) \\
Z_{2,\lambda_{odd}}(\tau) &= 124^4 40 q^{32} \eta^{24} G_4^{12} \big(760G_2E_2^3 - 540(G_2^2 + 3G_4)E_2^2 + 6(17G_2^3 + 411G_2G_4)E_2 \\
&\qquad - 11G_2^4 - 846G_2^2G_4 - 567G_4^2\big) \\
Z_{3,\lambda_{odd}}(\tau) &= 124^5 1120 q^{32} \eta^{24} G_4^{12} \big(36400G_2E_2^4 - 30240(G_2^2 + 3G_4)E_2^3 \\
&\quad + 840(11G_2^3 + 249G_2G_4)E_2^2 - 8(223G_2^4 + 17838G_2^2G_4 + 11907G_4^2)E_2 \\
&\quad + 3(29G_2^5 + 10206G_2^3G_4 + 30357G_2G_4^2)\big)
\end{aligned}
$$

Remark. Since the weights λ_{even} and λ_{odd} are primitive, we have

$$
Z_{g,\lambda_{even}}(q) = \sum_{h=0}^{g} C(h, g-h) \tilde{Z}_{h,\lambda_{even}}(q) \ ,
$$

and the corresponding formula for λ_{odd}.

3.3 BPS numbers $n_g(\beta)$. The BPS numbers $n_h(\beta)$ are related to Gromov-Witten invariants $N_g(\beta)$ by the formula (1.5). When $g = 0$ this formula reduces to $N_0(\beta) = \sum_{k|\beta} \frac{1}{k^3} n_0(\beta/k)$, which appeared in the original work by Candelas, de la Ossa, Green and Park [CdOGP] where it was found that $n_0(\beta)$ is integer-valued and interpreted as the number of rational curves of a fixed homology class β. When the rational curves are smooth and isolated, i.e. $O_C(-1) \oplus O_C(-1)$ curves in Calabi-Yau threefolds X, it is natural to have $n_0(\beta) = 1$, and in this case the multiple cover formula was proved in [AM][Ma]. Also, in this case, the higher genus generalization (1.5) was proved [FP] under further assumption that β is primitive. (In [FP], the formula (1.5) was proved also for the case β represents a super-rigid elliptic curve.)

Gopakumar-Vafa conjecture mentioned in section 1 contains a proposal for a "definition" of the number $n_h(\beta)$, which is independent to Gromov-Witten theory. The idea from string theory is that we may regard the number $n_g(\beta)$ as the number of BPS states of spin g and charge β in the context of M-theory. To describe its mathematical aspects briefly following [HST2], let X be a Calabi-Yau threefolds with an ample divisor L, and consider a moduli space $\mathcal{M}_\beta(X)$ of D2-branes, certain local systems supported on curves with homology class β. Under a suitable stability condition via L, the moduli space $\mathcal{M}_\beta(X)$ becomes projective. In [HST2], it is found that fixing the Hilbert polynomial to $P(m) = dm + 1$ $(d = \beta \cdot L)$ gives rise a moduli space consistent to the expectation from physics. We may consider a natural map $\pi_\beta : \mathcal{M}_\beta(X) \to Chow_\beta(X)$, where $Chow_\beta(X)$ is a subvariety in the Chow variety

$Chow_d(X)$ of degree d. Writing $S_\beta = \pi_\beta(\mathcal{M}_\beta(X))$ we have a surjective morphism $\pi_\beta : \mathcal{M}(X) \to S_\beta$. This is a brief sketch of the mathematical definitions made in [HST2] for the moduli spaces of D2 branes. Gopakumar and Vafa futher *expect* that there exist two Lefshetz sl_2's which act on the cohomology space $H^*(\mathcal{M}_\beta(X))$, one from the fiberwise Lefshetz action, denoted by $sl_{2,L}$, and the other from that of the base S_β, and denoted by $sl_{2,R}$. They also expects these two sl_2 commute and act on the E_2-term of the Leray spectral sequence. In [HST2], it has been pointed out that to ensure these sl_2 actions of the desired properties we need to use the Leray spectral sequence of the perverse sheaves[BBD] to the morphism $\mathcal{M}_\beta(X) \to S_\beta$. In this case, the sequence degenerates at the E_2-term and the two commuting sl_2 actions are realized in the interesection cohomology ring of $\mathcal{M}_\beta(X)$.

Assuming their existence, although the exsistence should be ensured as above, Gopakumar and Vafa has identified these two Lefschetz sl_2 actions on $H^*(\mathcal{M}_\beta(X))$ with the spin operators $SU(2)_L \times SU(2)_R$ acting on the BPS states in 5 dimensions. Then they introduce the following decomposition (in the representation ring);

$$H^*(\mathcal{M}_\beta(X)) = (I_0 \otimes R_0) \ \oplus \ (I_1 \otimes R_1) \ \oplus \ \cdots \ \oplus \ (I_g \otimes R_g) \tag{3.3}$$

where $I_h = \big((0) \oplus (\tfrac{1}{2})\big)^{\otimes h}$ is the $sl_{2,L}$ representation. The $sl_{2,R}$ representation $R_h (0 \leq h \leq g = g_\beta)$ should be understood as defined by the above decomposition. Then the invariants $n_h(\beta)$, which are integral by definition, are given by the "index";

$$n_h(\beta) = Tr_{R_h}(-1)^{H_R} \ \ , \tag{3.4}$$

where H_R is the generator of the Cartan subalgebra of $sl_{2,R}$ in Chevalley basis. This is the proposed "definition" of the number of BPS states of spin h and charge β. This proposed "definition" has been made mathematically more precise based on the definition $\mathcal{M}_\beta(X)$ and the Leray spectral sequence for perverse sheaves as described above. Based on this precise definition, the cases in which β represents a multiple of a rigid rational curve and also a multiple of a super-rigid elliptic curve E are studied in details, and consistent answers are obtained for $n_h(\beta)$. (See also [BP].) Also a closed formula [HST1, Proposition 1.2, 1.3] for $\sum_g Z_{g;1}(q)\lambda^{2g-2}$ has been proved by this precise definition [HST2, Theorem 4.10].

If we think that Gopakumar-Vafa conjecture, the formula (1.5), connects the BPS numbers defined above to Gromov-Witten invariants, the content of the conjecture becomes highly non-trivial as explained above. However, here in this paper, we simply list the results for $n_h(\beta)$ which results from Gromov-Witten invariants assuming the relation (1.5). In Table $2-5$, we have listed the numbers for $n_h(\beta)$ for $n = 1, 2, 3, 4$ and for each $W_{\hat{E}_8}$-orbits, (see section 3.1). As we see in our listing, the resulting BPS numbers are all integers supporting Gopakumar-Vafa conjecture. Furthermore we may interpret some of these numbers following the expected 'definition' (3.4) (, see Remark below).

To make our listing, we have to accomplish the orbit decompositions for higher levels (, $n = 3, 4$). Since the process is so technical, we omit the details here. But the idea is to use holomorphic anomaly equation (1.4) for other parametrizations $\mathcal{Z}_{g;n}(tD)$ with $D = H, e_9 + F, e_8 + e_9 + F, e_7 + e_8 + e_9 + F, e_6 + e_7 + e_8 + e_9 + F$, respectively, and make orbit decompositions for each. For example, the multiplicity function $P_{\lambda_0}(tD)$ may be determined to be

$$\Theta^{\mathbf{Z}}_{E_8}(t; 0^8) \ \ , \ \ \Theta^{\mathbf{Z}}_{E_8}(2t; t, t, 0^6) \ \ , \ \ \Theta^{\mathbf{Z}}_{E_8}(3t; 2t, t, t, 0^5) \ \ , \ \ \Theta^{\mathbf{Z}}_{E_8}(4t; 3t, t, t, t, 0^4) \ \ ,$$

respectively for $D = e_9+F, e_8+e_9+F, e_7+e_8+e_9+F, e_6+e_7+e_8+e_9+F$. The form $P_{\lambda_0}(tH) = \Theta^{\mathbf{Z}}_{E_8}(3t; t, \cdots, t, -t)$ was first appeared in [HSS]. These parametrizations have also been utilized in a recent work [Moh].

Remark. (1) As we observe in our Tables 2–5, the numbers $n_h(\beta)$ are *integral*. Similar observations are also made in [KZ][KKV] for several del Pezzo surfaces in Calabi-Yau threefolds. Since the Gromov-Witten invariants $N_g(\beta)$ are invariant under bi-rational transformations (if β does not intersect with the divisor of the bi-rational maps)[AGM], our $N_g(\beta)$ or $n_g(\beta)$ for rational elliptic surface S contain the corresponding invariants for all del Pezzo surfaces obtained by blowing up $k(\leq 9)$ points. For example, for the class $\beta = H = \lambda_8$ in Table 4 we see the genus zero invariants for (local) $\mathbf{P}^2$, i.e. $n_0(H)I_0 = 3I_0$. Also in Table 4, we see $n_0(\lambda_6)I_0 + n_1(\lambda_6)I_1 = 27I_0 - 4I_1$, i.e. the invariants for the del Pezzo surface Bl_6, see e.g. [KZ]. (Note that $\lambda_6 = (3; 1, 1, 1, 1, 1, 1, 0, 0, 0)$ may be read as the class of the anti-canonical bundle on the cubic surface.) In a similar way, we may continue our identification or interpretation of the numbers $n_h(\beta)$, although complete understanding of these numbers is beyond our scope of present paper. For the case of (local) $\mathbf{P}^2$, several numbers $n_h(d) := n_h(dH)$ has been verified in [KKV] under suitable 'understanding' of Gopakumar-Vafa conjecture (, see below).

(2) In ref.[GV], assuming the fibration $\mathcal{M}_\beta(X) \to S_\beta$ and the decomposition (3.3), it is argued in general that

$$n_0(\beta) = (-1)^{\dim \mathcal{M}_\beta(X)} \chi(\mathcal{M}_\beta(X)) \ , \ n_g(\beta) = (-1)^{\dim S_\beta} \chi(S_\beta) \ , \qquad (3.5)$$

where χ represents the Euler number. (These equations hold also in the formulation via intersection cohomology.) Also the $D2$ brane moduli space $\mathcal{M}_\beta(X)$ is naively claimed as the Jacobian fibration made over the moduli space of curves $C \subset X$ with $[C] = \beta$, which we write S_β. This description of the moduli space $\mathcal{M}_\beta(X)$ is to naive since there appears the cases of singular curves or even worse non-reduced curves in the family of the curves. However this naive definition provides 'nice' (although not quite correct in general) intuitions for the numbers $n_h(\beta)$. For example, the intuition about $n_0(\beta)$ is the Euler number of the locus for the nodal rational curves appears on S_β, which has been justified in [YZ][Be] for X =K3. This intuition is also naively expected in [GV] for general $n_h(\beta)$ $(0 \leq h \leq g)$, i.e. the numbers are the Euler numbers of the locus on S_β where nodal (genus h) curves appear. Again, because of the possible complicated degenerations of the curves, it is known that for this intuition to work, we need to take into account some corrections, by hand, depending on degeneration type[KKV].

In our case of curves in a surface S, the moduli space S_β of the curves may be understood as the linear system of the divisor class β identifying $H_2(S, \mathbf{Z})$ with $H^2(S, \mathbf{Z})$. Then the predicted numbers $n_g(\beta)$ in (3.5) is, up to sign, simply (the dimension of the linear system)+1, which we can verify all in our listing. In contrary to this, for the verification of $n_0(\beta) = (-1)^{\dim \mathcal{M}_\beta(X)} \chi(\mathcal{M}_\beta(X))$, we need more a precise definition of the moduli space. However if we restrict our attention to β's which give homology classes of elliptic curve in S, we may explain the numbers $n_0(\beta)$ from a naive definition of $\mathcal{M}_\beta(X)$ as the Jacobian fibration over the linear system S_β. The homology classes which admit this simple interpretation are;

$$\beta = 3H, 3H - e_1, 3H - e_1 - e_2, \cdots, 3H - e_1 - e_2 - \cdots - e_8 \ ,$$

for all of these we have the arithmetic genus 1. In fact, these classes may be regarded as the anti-canonical classes of respective del Pezzo surfaces Bl_k (k points blow up of $\mathbf{P}^2$) and therefore general elements of the linear system define an elliptic curve. We can find this kind of homology classes in our listing;

$$\begin{aligned} \lambda_0 + F = (3;1,1,1,1,1,1,1,1,0)\lambda_7 = (3;1,1,1,1,1,1,1,0,0) \\ \lambda_6 = (3;1,1,1,1,1,1,0,0,0)\lambda_5 = (3;1,1,1,1,1,0,0,0,0), \end{aligned} \tag{3.6}$$

and the corresponding numbers $n_0(\beta)I_0 + n_1(\beta)I_1$ are read, respectively, as

$$12I_0 - 2I_1 \ , \ -20I_0 + 3I_1 \ , \ 27I_0 - 4I_1 \ , \ -32I_0 + 5I_1. \tag{3.7}$$

The case $\beta = 3H$ is not contained in our listing, since it appears in $(\beta, F) = 9$, however, it is known the numbers are $27I_0 - 10I_1$ (see e.g. [KZ]). In all cases, the number $n_1(\beta)$ is given, up to sign, by the dimension of the linear system (plus one) considered in the respective del Pezzo surfaces, Bl_k $(k = 8,7,6,5,0)$. Also we may understand the numbers $n_0(\beta)$ following the argument given in [GV] for the case $\beta = 3H$. Namely, the naive moduli space $\mathcal{M}_\beta$ as the Jacobian fibration may be described by specifying a point on curves parametrized by the linear system. Since the specified point can move over the respective surface Bl_k $(k \le 7)$, this entails fibration $Bl_k \to \mathcal{M}_\beta \to \mathbf{P}^{\dim\,|\beta|-1}$. For $k = 8$ we need some special cares since the dimension of the linear system is one. However, for this case, from slightly different view point one may argue that $\mathcal{M}_\beta = \frac{1}{2}\mathrm{K3}$ (, see e.g. [HST1,2]). Evaluating the Euler number of $\mathcal{M}_\beta$, we obtain $n_0(\beta) = (-1)^{\dim \mathcal{M}_\beta}\chi(Bl_k)\dim\,|\beta|$ $(k \le 7)$. In this way we explain the numbers $n_0(\beta)$ in (3.7) as $12 = \chi(\frac{1}{2}\mathrm{K3}),\ -20 = -\chi(Bl_7) \times \chi(\mathbf{P}^1),\ 27 = \chi(Bl_6) \times \chi(\mathbf{P}^2),\ -32 = -\chi(Bl_5) \times \chi(\mathbf{P}^3)$.

Some detailed arguments may be found in [KKV] to 'explain' the numbers $n_h(\beta)$ as the Euler numbers with some corrections of the degeneration locus of curves on S_β. Following the arguments there, we may understand some other numbers $n_h(\beta)$ in our tables. Recently it is announced to the author that for several β in the Table 3–5, we can verify $n_h(\beta)$ following the definition given in [HST2], i.e. from the definition of $\mathcal{M}_\beta(X)$ given there, the Leray spectral sequence of the perverse sheaves and the intersection cohomology[Ta](, Table 2 has been proved in [HST2, Theorem 4.10]). However we still do not have full geometrical verifications of these integer numbers $n_h(\beta)$ presented in Table 3–5.

β	$\sum_g n_g(\beta) I_g$ for $\lambda_0 + a\,F = (0,0,0,0,0,0,0,0,0,-1) + a\,F$							
λ_0	$1I_0$							
$\lambda_0 + F$	$12I_0$	$-2I_1$						
$\lambda_0 + 2F$	$90I_0$	$-30I_1$	$+3I_2$					
$\lambda_0 + 3F$	$520I_0$	$-260I_1$	$+52I_2$	$-4I_3$				
$\lambda_0 + 4F$	$2535I_0$	$-1690I_1$	$+507I_2$	$-78I_3$	$+5I_4$			
$\lambda_0 + 5F$	$10908I_0$	$-9090I_1$	$+3636I_2$	$-840I_3$	$+108I_4$	$-6I_5$		
$\lambda_0 + 6F$	$42614I_0$	$-42614I_1$	$+21307I_2$	$-6570I_3$	$+1271I_4$	$-142I_5$	$+7I_6$	
$\lambda_0 + 7F$	$153960I_0$	$-179620I_1$	$+107772I_2$	$-41580I_3$	$+10756I_4$	$-1812I_5$	$+180I_6$	$-8I_7$

Table 2. BPS numbers for $\beta = e_9 + a\,F$ ($a \leq 7$). A closed formula valid for all $a \geq 0$ is known in [HST1].

β	$\sum_g n_g(\beta) I_g$ for $\beta = 2\lambda_0 + a\,F = (6,2,2,2,2,2,2,2,2,0) + a\,F$										
$2\lambda_0$	$-132I_0$	$+42I_1$	$-4I_2$								
$2\lambda_0 + F$	$-3680I_0$	$+2280I_1$	$-644I_2$	$+96I_3$	$-6I_4$						
$2\lambda_0 + 2F$	$-60120I_0$	$+56430I_1$	$-26658I_2$	$+7904I_3$	$-1492I_4$	$+164I_5$	$-8I_6$				
$2\lambda_0 + 3F$	$-715968I_0$	$+901008I_1$	$-599080I_2$	$+267340I_3$	$-84538I_4$	$+18772I_5$	$-2768I_6$	$+248I_7$	$-10I_8$		
$2\lambda_0 + 4F$	$-6854200I_0$	$+10830300I_1$	$-9294204I_2$	$+5549948I_3$	$-2469482I_4$	$+829340I_5$	$-207648I_6$	$+37560I_7$	$-4632I_8$	$+348I_9$	$-12I_{10}$
β	$\sum_g n_g(\beta) I_g$ for $\beta = \lambda_1 + a\,F = (1,1,0,0,0,0,0,0,0,0) + a\,F \quad (\lambda_1 = \lambda_{even})$										
λ_1	$-2I_0$										
$\lambda_1 + F$	$-144I_0$	$+44I_1$	$-4I_2$								
$\lambda_1 + 2F$	$-3760I_0$	$+2332I_1$	$-654I_2$	$+96I_3$	$-6I_4$						
$\lambda_1 + 3F$	$-60480I_0$	$+56752I_1$	$-26784I_2$	$+7920I_3$	$-1492I_4$	$+164I_5$	$-8I_6$				
$\lambda_1 + 4F$	$-717552I_0$	$+903068I_1$	$-600186I_2$	$+267620I_3$	$-84566I_4$	$+18772I_5$	$-2786I_6$	$+248I_7$	$-10I_8$		
$\lambda_1 + 5F$	$-6860128I_0$	$+10839688I_1$	$-9301032I_2$	$+5552504I_3$	$-2469980I_4$	$+829380I_5$	$-207648I_6$	$+37560I_7$	$-4632I_8$	$+348I_9$	$-12I_{10}$
β	$\sum_g n_g(\beta) I_g$ for $\beta = \lambda_7 + a\,F = (3,1,1,1,1,1,1,1,0,0) + a\,F \quad (\lambda_7 = \lambda_{odd})$										
λ_7	$-20I_0$	$+3I_1$									
$\lambda_7 + F$	$-792I_0$	$+366I_1$	$-68I_2$	$+5I_3$							
$\lambda_7 + 2F$	$-15768I_0$	$+12282I_1$	$-4620I_2$	$+1022I_3$	$-128I_4$	$+7I_5$					
$\lambda_7 + 3F$	$-214848I_0$	$+235952I_1$	$-134072I_2$	$+49705I_3$	$-12528I_4$	$+2076I_5$	$-204I_6$	$+9I_7$			
$\lambda_7 + 4F$	$-2270340I_0$	$+3221991I_1$	$-2452812I_2$	$+1278828I_3$	$-486344I_4$	$+135545I_5$	$-26992I_6$	$+3634I_7$	$-296I_8$	$+11I_9$	

Table 3. BPS numbers for $n_g(\beta)$ with $(\beta, F) = 2$ up to genus 10.

β	$\sum_g n_g(\beta) I_g$ for the orbit $\beta = \lambda_8 + a\,F = (1,0,0,0,0,0,0,0,0,0) + a\,F$										
λ_8	$3I_0$										
$\lambda_8 + F$	$+1005I_0$	$-456I_1$	$+84I_2$	$-6I_3$							
$\lambda_8 + 2F$	$+73374I_0$	$-67566I_1$	$+31533I_2$	$-9216I_3$	$+1713I_4$	$-186I_5$	$+9I_6$				
$\lambda_8 + 3F$	$+2697432I_0$	$-3759336I_1$	$+2827800I_2$	$-1458114I_3$	$+548313I_4$	$-151248I_5$	$+29868I_6$	$-3996I_7$	$+324I_8$	$-12I_9$	
β	$\sum_g n_g(\beta) I_g$ for the orbit $\beta = \lambda_6 + a\,F = (3,1,1,1,1,1,1,0,0,0) + a\,F$										
λ_6	$+27I_0$	$-4I_1$									
$\lambda_6 + F$	$+4644I_0$	$-2826I_1$	$+786I_2$	$-114I_3$	$+7I_4$						
$\lambda_6 + 2F$	$+258390I_0$	$-278532I_1$	$+156447I_2$	$-57348I_3$	$+14279I_4$	$-2340I_5$	$+228I_6$	$-10I_7$			
$\lambda_6 + 3F$	$+8103780I_0$	$-12578006I_1$	$+10664253I_2$	$-6297870I_3$	$+2771484I_4$	$-921480I_5$	$+228837I_6$	$-41134I_7$	$+5049I_8$	$-378I_9$	$+13I_{10}$
β	$\sum_g n_g(\beta) I_g$ for the orbit $\beta = \lambda_0 + \lambda_1 + a\,F = (4,2,1,1,1,1,1,1,1,0) + a\,F$										
$\lambda_0 + \lambda_1$	$+180I_0$	$-54I_1$	$+5I_2$								
$\lambda_0 + \lambda_1 + F$	$+19242I_0$	$-14706I_1$	$+5477I_2$	$-1198I_3$	$+148I_4$	$-8I_5$					
$\lambda_0 + \lambda_1 + 2F$	$+856368I_0$	$-1058112I_1$	$+695074I_2$	$-306578I_3$	$+95784I_4$	$-21042I_5$	$+3097I_6$	$-274I_7$	$+11I_8$		
β	$\sum_g n_g(\beta) I_g$ for the orbit $\beta = \lambda_0 + \lambda_7 + a\,F = (6,2,2,2,2,2,2,2,1,0) + a\,F$										
$\lambda_0 + \lambda_7$	$+927I_0$	$-420I_1$	$+79I_2$	$-6I_3$							
$\lambda_0 + \lambda_7 + F$	$+72288I_0$	$-66552I_1$	$+31126I_2$	$-9134I_3$	$+1706I_4$	$-186I_5$	$+9I_6$				
$\lambda_0 + \lambda_7 + 2F$	$+2686608I_0$	$-3744072I_1$	$+2817609I_2$	$-1454040I_3$	$+547302I_4$	$-151104I_5$	$+29859I_6$	$-3996I_7$	$+324I_8$	$-12I_9$	
β	$\sum_g n_g(\beta) I_g$ for the orbit $\beta = 3\lambda_0 + a\,F = (9,3,3,3,3,3,3,3,3,0) + a\,F$										
$3\lambda_0$	$+4068I_0$	$-2468I_1$	$+704I_2$	$-108I_3$	$+7I_4$						
$3\lambda_0 + F$	$+251235I_0$	$-270720I_1$	$+152607I_2$	$-56314I_3$	$+14133I_4$	$-2332I_5$	$+228I_6$	$-10I_7$			
$3\lambda_0 + 2F$	$+8037792I_0$	$-12474432I_1$	$+10585764I_2$	$-6261210I_3$	$+2760362I_4$	$-919362I_5$	$+228611I_6$	$-41124I_7$	$+5049I_8$	$-378I_9$	$+13I_{10}$

Table 4. BPS numbers for $n_g(\beta)$ with $(\beta, F) = 3$ up to genus 10.

β	$\sum_g n_g(\beta) I_g$ for $\beta = \lambda + a\,F$ ($\lambda \in \mathcal{P}_{+,4}^{min}$)										
λ_2	$-4I_0$										
$\lambda_2 + F$	$-5080I_0$	$+3152I_1$	$-900I_2$	$+132I_3$	$-8I_4$						
$\lambda_2 + 2F$	$-911732I_0$	$+1137736I_1$	$-758204I_2$	$+337896I_3$	$-105996I_4$	$+23256I_5$	$-3408I_6$	$+300I_7$	$-12I_8$		
λ_5	$-32I_0$	$+5I_1$									
$\lambda_5 + F$	$-20736I_0$	$+16104I_1$	$-6128I_2$	$+1358I_3$	$-168I_4$	$+9I_5$					
$\lambda_5 + 2F$	$-2856896I_0$	$+4016110I_1$	$-3060712I_2$	$+1593888I_3$	$-602104I_4$	$+166083I_5$	$-32704I_6$	$+4358I_7$	$-352I_8$	$+13I_9$	
$\lambda_1 + \lambda_7$	$-200I_0$	$+62I_1$	$-6I_2$								
$\lambda_1 + \lambda_7 + F$	$-77936I_0$	$+72732I_1$	$-34608I_2$	$+10258I_3$	$-1918I_4$	$+208I_5$	$-10I_6$				
$\lambda_1 + \lambda_7 + 2F$	$-8562584I_0$	$+13391466I_1$	$-11489290I_2$	$+6847470I_3$	$-3026698I_4$	$+1006676I_5$	$-249436I_6$	$+44686I_7$	$-5466I_8$	$+408I_9$	$-14I_{10}$
$\lambda_0 + \lambda_8$	$-1056I_0$	$+491I_1$	$-94I_2$	$+7I_3$							
$\lambda_0 + \lambda_8 + F$	$-273472I_0$	$+298178I_1$	$-170310I_2$	$+63215I_3$	$-15832I_4$	$+2595I_5$	$-252I_6$	$+11I_7$			
$2\lambda_1$	$-1120I_0$	$+520I_1$	$-100I_2$	$+7I_3$							
$2\lambda_1 + F$	$-276416I_0$	$+301376I_1$	$-172104I_2$	$+63814I_3$	$-15944I_4$	$+2604I_5$	$-252I_6$	$+11I_7$			
$2\lambda_7$	$-976I_0$	$+453I_1$	$-88I_2$	$+7I_3$							
$2\lambda_7 + F$	$-269952I_0$	$+294304I_1$	$-168272I_2$	$+62586I_3$	$-15720I_4$	$+2586I_5$	$-252I_6$	$+11I_7$			
$\lambda_0 + \lambda_6$	$-4656I_0$	$+2888I_1$	$-830I_2$	$+125I_3$	$-8I_4$						
$\lambda_0 + \lambda_6 + F$	$-896544I_0$	$+1118712I_1$	$-746184I_2$	$+333193I_3$	$-104840I_4$	$+23093I_5$	$-3398I_6$	$+300I_7$	$-12I_8$		
$2\lambda_0 + \lambda_1$	$-18496I_0$	$+14358I_1$	$-5506I_2$	$+1247I_3$	$-160I_4$	$+9I_5$					
$2\lambda_0 + \lambda_1 + F$	$-2788864I_0$	$+3920112I_1$	$-2991232I_2$	$+1561730I_3$	$-592212I_4$	$+164132I_5$	$-32482I_6$	$+4347I_7$	$-352I_8$	$+13I_9$	
$2\lambda_0 + \lambda_7$	$-66912I_0$	$+62408I_1$	$-29968I_2$	$+9083I_3$	$-1758I_4$	$+199I_5$	$-10I_6$				
$2\lambda_0 + \lambda_7 + F$	$-8268048I_0$	$+12929124I_1$	$-11111688I_2$	$+6645513I_3$	$-2952212I_4$	$+987939I_5$	$-246382I_6$	$+44397I_7$	$-5454I_8$	$+408I_9$	$-14I_{10}$
$4\lambda_0$	$-224688I_0$	$+244793I_1$	$-141352I_2$	$+53760I_3$	$-13956I_4$	$+2387I_5$	$-242I_6$	$+11I_7$			

The minimal dominant integral weights in $\mathcal{P}_{+,4}^{min}$:

$\lambda_2 = (2;1,1,0,0,0,0,0,0,0)$ $\quad \lambda_0+\lambda_8 = (4;1,1,1,1,1,1,1,1,0)$ $\quad \lambda_0+\lambda_6 = (6;2,2,2,2,2,2,1,1,0)$ $\quad 4\lambda_0 = (12;4,4,4,4,4,4,4,4,0)$

$\lambda_5 = (3;1,1,1,1,1,0,0,0,0)$ $\quad 2\lambda_1 = (5;3,1,1,1,1,1,1,1,1)$ $\quad 2\lambda_0+\lambda_1 = (7;3,2,2,2,2,2,2,2,0)$

$\lambda_1+\lambda_7 = (4;2,1,1,1,1,1,1,0,0)$ $\quad 2\lambda_7 = (6;2,2,2,2,2,2,2,0,0)$ $\quad 2\lambda_0+\lambda_7 = (9;3,3,3,3,3,3,3,2,0)$

Table 5. BPS numbers for $n_g(\beta)$ with $(\beta, F) = 4$ up to genus 10.

4 Bershadsky-Cecotti-Ooguri-Vafa holomorphic anomaly equation

In previous sections, we have analyzed the holomorphic anomaly equation (1.4) of rational elliptic surface in detail. Here we continue our analysis based on Bershadsky-Ceccoti-Ooguri-Vafa (BCOV) holomorphic anomaly equation. BCOV holomorphic anomaly equation is a general formula for partition functions of the topological sigma model with target space Calabi-Yau 3-folds. Therefore it is applicable, in principle, for general Calabi-Yau 3-folds to determine the higher genus prepotential $\mathtt{F}_g$. However, unfortunately, solving the equation is so complicated that Calabi-Yau models for which we can determine $\mathtt{F}_g$ are very restricted (, e.g. in references [BCOV2][KKV] $\mathtt{F}_g$ up to $g=5$ has been analyzed only for those models of one dimensional moduli of Kähler deformation, i.e $rkH^2(X,\mathbf{Z})=1.$). In a recent paper [KZ] it has been found that a considerable simplification occurs in the local mirror limit finding that the dilaton does not propagate under this limit. Using this fact prepotentials $\mathtt{F}_g$ $(g\leq 8)$ have been determined for rational surfaces, p-points blow up of $\mathbf{P}^2$ $(0\leq p\leq 8)$ and $\mathbf{P}^1\times\mathbf{P}^1$, restricting the deformation parameter to a specific direction. Although we see considerable simplification in the local mirror limit, the higher genus calculations are still tedious because of formidable growth of graphs we need to sum up.

In this section we will analyze the local limit of BCOV holomorphic anomaly equation for 12K3, realizing the surface as a smooth divisor in a Calabi-Yau threefold. The aims of this section are two-folded; the first is to see a consistency between our equation (1.4) and BCOV holomorphic anomaly equation. As we will see in the following, they produce the same results although their equivalence seems non-trivial. The second is to show examples of two parameter deformations for which we can still manipulate BCOV holomorphic anomaly equation.

Recently many progresses have been made in counting holomorphic discs, so-called disc instantons, with their boundary on a Lagrangian submanifold in (non-compact) Calabi-Yau threefolds. See references [OV],[AV],[AVK],[LM], and also [GZ],[LK], [LLY2] for suitable extension of the moduli space of stable maps to disc instantons. Most recently, very non-trivial relations to Chern-Simons gauge theory which enables us to write down all genus generating function has been found in [AMV],[DFG](, e.g. Table 6 in [AMV] exactly coincides with our Table 8 below). In this paper, however, our attention will be restricted to the case of *old* instantons.

4.1 BCOV holomorphic anomaly equation. In the original paper by Bershadsky, Cecotti, Ooguri and Vafa [BCOV1], the higher genus prepotential $\mathcal{F}_g$ has been defined as a partition function of the topological sigma model with its target space Calabi-Yau 3 fold X and the world sheet being genus g Riemann surfaces. $\mathcal{F}_g$ is expected to be a holomorphic function (section) on the moduli space of Calabi-Yau manifolds after the topological twist, however, they found that there is *holomorphic anomaly*. To describe it very briefly, let us consider a Calabi-Yau threefold X, and denote its mirror Calabi-Yau threefold by $X^\vee$. We consider its (local) deformation family $\{X^\vee_x\}_{x\in\mathcal{M}^0(X^\vee)}$ writing the deformation space by $\mathcal{M}^0(X^\vee)$. (We are mainly interested in a local deformations near so called *large complex structure limit*, where the monodromy become maximally degenerated.) Since the deformations are unobstracted [Ti],[To],[Bo], we may assume $\mathcal{M}^0(X^\vee)$ is smooth, and introduce Weil-Peterson metric by the Kähler potential $K(x,\bar{x})$ with $e^{-K}=\int_{X^\vee}\bar{\Omega}_x\wedge\Omega_x$ where Ω_x is the nowhere vanishing holomorphic 3-from of $X^\vee_x$ $(x\in\mathcal{M}^0(X^\vee))$. We may assume a compactified complex structure moduli space $\mathcal{M}^{cpl}(X^\vee)$ in some sense, which naturally exists, e.g. for monomial deformations of hypersurfaces, and may consider the Kähler geometry patching the above local geometry on $\mathcal{M}^{cpl}(X^\vee)$.

Let us denote by $\mathcal{L}$ the holomorphic line bundle on $\mathcal{M}^{cpl}(X^\vee)$ whose section is given by Ω_x. Then $e^{-K(x,\bar{x})}$ is a section of $\mathcal{L}\otimes\bar{\mathcal{L}}$. Also we may consider the Griffith-Yukawa coupling $C_{ijk} := -\int_{X^\vee}\Omega_x\wedge\partial_{x_i}\partial_{x_j}\partial_{x_k}\Omega_x$ and its complex conjugate $C_{\bar{i}\bar{j}\bar{k}} := \overline{C_{ijk}}$, which are regarded as a section of $\mathcal{L}^{\otimes 3}$ and $\bar{\mathcal{L}}^{\otimes 3}$, respectively. BCOV identifies the higher genus prepotential $\mathcal{F}_g$ as an almost holomorphic section of $\mathcal{L}^{2-2g}$ but with holomorphic anomaly described by

$$\partial_{\bar{x}_i}\mathcal{F}_g = \frac{1}{2}e^{2K}\sum_{j,k,\bar{j},\bar{k}} C_{\bar{i}\bar{j}\bar{k}}G^{j\bar{j}}G^{k\bar{k}}\Big(\sum_{r=0}^{g} D_j\mathcal{F}_r D_k\mathcal{F}_{g-r} + D_jD_k\mathcal{F}_{g-1}\Big)\,, \tag{4.1}$$

where $G^{i\bar{j}}$ is the inverse of the Weil-Peterson metric $G_{i\bar{j}} = \partial_{x_i}\partial_{\bar{x}_j}K(x,\bar{x})$ and $D_j : T^{1,0}\mathcal{M}^{cpl}(X^\vee)\otimes\mathcal{L}^{\otimes n} \rightarrow T^{1,0}\mathcal{M}^{cpl}(X^\vee)\otimes\mathcal{L}^{\otimes n}$ is the covariant derivative, which acts on a vector field Z^k taking value on $\mathcal{L}^{\otimes n}$ by $D_jZ^k = \partial_{x_j}Z^k + \sum_l \Gamma^k_{jl}Z^l - n\partial_{x_j}KZ^k$ where Γ^k_{jl} is the metric connection. As we see here, the holomorphic anomaly equation (1.4) is very similar to BCOV holomorphic anomaly equation. They share similar forms, however, associated meaning seems to be slightly different. For example, in the case of BCOV equation (4.1), the holomorphic ambiguity arises from the nontrivial holomorphic sections of $\mathcal{L}^{2-2g}$, which we write hereafter $f_g(x) \in H^0(\mathcal{M}^{cal}(X^\vee), \mathcal{L}^{2-2g})$. In the end of this subsection, we will compare this holomorphic ambiguity with that of $f_{2g+6n-2}(E_4,E_6)$ for (1.4).

In [BCOV1,2], the general solution of the the holomorphic anomaly equation (4.1) has been constructed using the Kähler geometry (, more precisely *special Kähler geometry*,) on the moduli space $\mathcal{M}^{cpl}(X^\vee)$. There it was also found that the solutions give the generating functions of higher genus Gromov-Witten invariants, $\mathtt{F}_g(t) = \sum_\beta N_g(\beta)q^\beta$ $(q^{2\pi\sqrt{-1}t})$. Namely it is claimed that when we introduce the flat coordinate $t_i = t_i(x)$ characterized by $\Gamma^{t_k}_{t_it_j} = 0$ and a property $t_i \sim \frac{1}{2\pi\sqrt{-1}}log\, x_i$ near the large complex structure limit point, then the generating functions will be given by $\mathtt{F}_g(t) := (w_0(x))^{2g-2}\mathcal{F}_g(x)$. Where $w_0(x)$ is the unique period integral which is regular at the large complex structure limit point and behaves like $w_0(x) = 1 + O(x)$ near that point. General recursive formula valid for all genera may be found in [BCOV2], however for simplicity, here we only reproduce their results for the case of genus two.

(Solution of BCOV holomorphic anomaly equation at $g=2$*) Assume the generating functions* $\mathtt{F}_0(t)$ *and* $\mathtt{F}_1(t)$ *are determined. Then there exist propagaters* $\mathtt{S}^{t_it_j}, \mathtt{S}^{t_i\phi}, \mathtt{S}^{\phi\phi}$ *(symmetric tensors on* $\mathcal{M}^{cal}(X^\vee)$*), and a holomorphic section* $f_2(x)$ *of* $\mathcal{L}^{\otimes 2}$ *which express the genus generating function* $\mathtt{F}_2(t)$ *by*

$$\begin{aligned}\mathtt{F}_2(t) = 12\sum_{j,k}\mathtt{S}^{t_it_j}\left(\partial_{t_j}\partial_{t_k}\mathtt{F}_1 + \partial_{t_j}\mathtt{F}_1\partial_{t_k}\mathtt{F}_1\right)\\ -14\sum_{j,k,m,n}\mathtt{S}^{t_jt_k}\mathtt{S}^{t_mt_n}\left(12\mathtt{K}_{t_jt_kt_mt_n} + 2\mathtt{K}_{t_mt_nt_j}\partial_{t_k}\mathtt{F}_1\right) + \chi 24\sum_k \mathtt{S}^{t_k\phi}\partial_{t_k}\mathtt{F}_1\end{aligned}$$

$$
\begin{aligned}
&+18\sum_{j,k,m,n,r,s}\mathtt{S}^{t_jt_k}\mathtt{S}^{t_mt_n}\mathtt{S}^{t_rt_s}\mathtt{K}_{t_mt_nt_j}\mathtt{K}_{t_kt_rt_s}\\
&+112\sum_{a,b,j,k,m,n}\mathtt{K}_{t_at_jt_m}\mathtt{K}_{t_bt_kt_n}\mathtt{S}^{t_at_b}\mathtt{S}^{t_jt_k}\mathtt{S}^{t_mt_n} \qquad (4.2)\\
&-\chi 48\mathtt{S}^{t_j\phi}\mathtt{S}^{t_mt_n}\mathtt{K}_{t_mt_nt_j}+\chi 24(\chi 24-1)\mathtt{S}^{\phi\phi}+w_0(x)^2 f_2(t)\ ,
\end{aligned}
$$

where $\chi=\chi(X)$ is the Euler number of X and ϕ is called dilaton. Also we set, three point and four point functions, respectively, by

$$
\mathtt{K}_{t_mt_nt_j}:=\partial_{t_m}\partial_{t_n}\partial_{t_j}\mathtt{F}_0(t)\ ,\ \mathtt{K}_{t_mt_nt_jt_k}:=\partial_{t_m}\partial_{t_n}\partial_{t_j}\partial_{t_k}\mathtt{F}_0(t)\ .
$$

Remark. (1) In the above formula, the propagater $\mathtt{S}^{t_it_j}$ is determined by solving relation $\sum_m \mathtt{K}_{t_it_jt_m}\mathtt{S}^{t_mt_k}=\partial_{t_i}K\delta^{t_k}_{t_j}+\partial_{t_j}K\delta^{t_k}_{t_i}-\Gamma^{t_k}_{t_it_j}$, which arises from special Kähler geometry on $\mathcal{M}^{cpl}(X^\vee)$. Other propagaters $\mathtt{S}^{t_i\phi}$ and $\mathtt{S}^{\phi\phi}$ are also determined by similar relations. Determining these propagaters is one of the most difficult parts to construct the solutions. Once these are determined, $\mathtt{F}_g(t)(g\geq 2)$ are determined summing over several terms which are in 1 to 1 corresponding to the graphs representing degenerations of genus g curves (see [BCOV1,2]). For each genus, we have to fix the holomorphic ambiguity $f_g(x)$ by vanishing conditions like those discussed in section 3.1.
(2) The flat coordinate $t_i=t_i(x)$ is called *mirror map*. It relates the complex structure moduli space of $X^\vee$ to the complexified Kähler cone $H^2(X,\mathbf{R})+\sqrt{-1}\mathcal{K}_X$. Then by the coordinate $(t_1,\cdots,t_r)$, we understand a point $\sum_i t_kJ_k\in H^2(X,\mathbf{R})+\sqrt{-1}\mathcal{K}_X$ with some positive integral generators $J_1,\cdots,J_r$ of $H^2(X,\mathbf{Z})$. See e.g. [HLY] for details. When some of the integral generators, say J_r, represents Poincaré dual of a smooth divisor S (with $K_S>0$), then the limit $Im(t_r)\to\infty$ is called *local mirror symmetry limit*. Projective space $\mathbf{P}^2$, del Pezzo surfaces (and also rational elliptic surfaces) as smooth divisor in Calabi-Yau threefolds are well-studied examples (see [CKYZ]).

As remarked above, constructing solutions of BCOV holomorphic anomaly equation involves three steps; 1) finding the propagaters, 2) summing over graphs parametrizing the degeneration, 3) fixing the holomorphic ambiguity. Since all of them are technically so involved that it is very hard to make solutions $\mathtt{F}_g$ in general. However it has been found in [KZ] that under *local mirror symmetry limit* the solutions for $\mathtt{F}_g$ are considerably simplified.

(Local mirror symmetry limit [KZ]) Under the local mirror symmetry limit to a smooth divisor, if it exists, the both propagaters $\mathtt{S}^{t_i\phi}$ and $\mathtt{S}^{\phi\phi}$ vanish. In other words, the dilaton ϕ does not propagate.

As we see in the genus two example (4.2), the local limit simplifies the form of $\mathtt{F}_g$. However its manipulation is still tedious unless $\mathtt{S}^{t_it_j}=S_i\delta^{t_it_j}$. For the local mirror symmetry limit to a rational elliptic surface S, our observation is that the simplification $\mathtt{S}^{t_it_j}=S_i\delta^{t_it_j}$ in fact occurs!

4.2 $\mathbf{S}=\frac{1}{2}\mathbf{K3}$. Here we present the form of the propagater $\mathtt{S}^{t_it_j}$ for rational elliptic surfaces, i.e. $S=\frac{1}{2}K3$. The main observation is the compatibility of the holomorphic anomaly equation (1.4) studied in detail in section 3 with the recursion relation (4.2) which follows from BCOV holomorphic anomaly equation.

Definition 4.1. Let $Z_{g;n}(q)$ $(q = e^{2\pi\sqrt{-1}\tau})$ be the solutions of the holomorphic anomaly equation (1.4). Then we define a series

$$\mathtt{F}_g(q,p) := \sum_{n\geq 1} Z_{g;n} p^n \quad . \tag{4.3}$$

Now let us introduce the following hypergeometric series:

$$w_0(x,y) := \sum_{n,m\geq 0} c(n,m) x^n y^m$$

$$c(n,m) := \frac{\Gamma(1+6n)}{\Gamma(1+3n)\Gamma(1+2n)\Gamma(1+n-m)\Gamma(1+m)^2\Gamma(1-m)}$$

The mirror map or the flat coordinate t_1, t_2 may be defined by this hypergeometric series;

$$t_i := \frac{1}{2\pi\sqrt{-1}} \frac{\partial_{\rho_i} w_0(x,y,\rho_1,\rho_2)}{w_0(x,y)}|_{\rho_i=0} \ ,$$

where $w_0(x,y,\rho_1,\rho_2) := \sum_{n,m\geq 0} c(n+\rho_1, m+\rho_2) x^{n+\rho_1} y^{m+\rho_2}$. We denote the inverse relation of $t_i = t_i(x,y)$ $(i=1,2)$ by $x = x(q,p), y = y(q,p)$ setting $q = e^{2\pi\sqrt{-1}t_1}, p = e^{2\pi\sqrt{-1}t_2}$. Detailed analysis of the mirror map can be found in [HST1], and following the method there it is straightforward to see $x = x(q)$ and $y = y(q,p)$, i.e. the relations are lower triangular. Furthermore it is easy to derive

$$x(q)(1-432x(q)) = \frac{1}{j(q)} \ \ , \ \ w_0(x(q), y(q,p)) = w_0(x(q)) = E_4(q)^{\frac{1}{4}} \ \ , \tag{4.4}$$

where $j(q)$ is the elliptic modular function and $w_0(x,y) = w_0(x)$ from the definition. The next statement follows directly from the derivation of the holomorphic anomaly equation given in [HST1], changing the parametrization there in an obvious way.

Proposition 4.2. *The functions $\mathtt{F}_0(q,p)$ and $\mathtt{F}_1(q,p)$ defined above may be written by the hypergeometric series $w_0(x,y), \partial_{\rho_1} w_0(x,y), \partial_{\rho_2} w_0(x,y)$ and $\partial_{\rho_1}\partial_{\rho_2} w_0(x,y)$. Especially $\mathtt{F}_1(q,p)$ is given by*

$$\mathtt{F}_1(q,p) = \frac{1}{2}\log\left\{ [(1-432x(1-y))(1-y)]^{-\frac{1}{6}} \frac{\partial y}{\partial t_2} \right\}\Big|_{x=x(q),y=y(q,p)} \ \ , \tag{4.5}$$

where $(1-432x(1-y))(1-y) =: dis_0$ is a component of the discriminant, which follows from the characteristic variety of the differential equation satisfied by the hypergeometric series.

Remark. (1) The discriminant from the differential equation may be found to be $xy(1-432x)^3 dis_0$, where the normal crossing divisors $x=0$ and $y=0$ give rise to the large complex structure limit.
(2) As is evident from the context, the flat coordinate t_1 should be identified with the modular parameter τ in (1.4). Then the holomorphic anomaly (or modular anomaly) in (1.4) comes from the 'anomalous' modular transformation;

$$E_2(\tau)\big|_{\tau\to\frac{a\tau+b}{c\tau+d}} = (c\tau+d)^2 E_2(\tau) + \frac{12}{2\pi\sqrt{-1}} c(c\tau+d),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element in $PSL(2,\mathbf{Z})$. As we have $x = x(q) = \frac{1\pm\sqrt{1-1728/j(q)}}{864}$ which is modular function (for a modular subgroup of index two), the modular anomaly should be traced to the form $y = y(q,p)$. Following exactly the same calculations presented in [HST1], we can determine $E_2(q)$-dependence of $y(q,p)$, and from which we can derive

$$y(q,p)\big|_{t_1\to\frac{at_1+b}{ct_1+d}} = y(q,p)e^{-\frac{1}{2\pi\sqrt{-1}}c(ct_1+d)\partial_{t_2}\mathtt{F}_0(q,p)} \quad . \tag{4.6}$$

Using this relation essentially, we can prove that $\mathtt{F}_1(q,p)$ given in (4.5) in fact satisfies the holomorphic anomaly equation (1.4) with $g=1$.

Now for our higher genus function $\mathtt{F}_g(q,p)$ ($g\geq 2$), we may observe the following:

Conjecture 4.3. *Define the propagater* $\mathtt{S}^{t_it_j}$ *by* $\mathtt{S}^{t_1t_1}=\mathtt{S}^{t_1t_2}=\mathtt{S}^{t_2t_1}=0$ *and*

$$\mathtt{S}^{t_2t_2} = -1\mathtt{K}_{t_2t_2t_2}\frac{\partial}{\partial t_2}\log\left(y\frac{\partial y}{\partial t_2}\right) \quad ,$$

and also $\mathtt{K}_{t_2t_2t_2}:=\partial_{t_2}\partial_{t_2}\partial_{t_2}\mathtt{F}_0(q,p)$. *Then there exists a rational function* $f_g(x,y)$ *of the form*

$$f_g(x,y) = (\textit{polynomial in } x,\, y)/(dis_0)^{2g-2},$$

which reproduces our function $\mathtt{F}_g(q,p)$ *in (4.3) from the BCOV recursion relation with vanishing dilaton propagaters (e.g. the recursion formula (4.2) for* $g=2$ *with* $\mathtt{S}^{t_i\phi}=\mathtt{S}^{\phi\phi}=0$. *)*

For example, for $g=2$ we have the reduced BCOV recursion relation,

$$\begin{aligned}\mathtt{F}_2(q,p) =& 12\mathtt{S}^{t_2t_2}\left(\partial_{t_2}\partial_{t_2}\mathtt{F}_1+\partial_{t_2}\mathtt{F}_1\partial_{t_2}\mathtt{F}_1\right) - 18\mathtt{S}^{t_2t_2}\mathtt{S}^{t_2t_2}\left(\mathtt{K}_{t_2t_2t_2t_2}+4\mathtt{K}_{t_2t_2t_2}\partial_{t_2}\mathtt{F}_1\right)\\ &+524\mathtt{S}^{t_2t_2}\mathtt{S}^{t_2t_2}\mathtt{S}^{t_2t_2}\mathtt{K}_{t_2t_2t_2}\mathtt{K}_{t_2t_2t_2} + w_0^2(q)f_2(q,p)\ ,\end{aligned}$$

with the holomorphic ambiguity $f_2(x,y)$. We may verify directly that our functions $\mathtt{F}_0,\mathtt{F}_1$ and $\mathtt{F}_2$ satisfy the above recursion relation with

$$\begin{aligned}&f_2(x,y) = 1\Big/\Big(240(1-432x(1-y))^2(1-y)^2\Big)\times\\ &\Big(\left(1-72x-311040x^2+67184640x^3\right)y+1430\left(x-1296x^2+373248x^3\right)y^2\\ &+2\left(751x-1386720x^2+599063040x^3\right)y^3+1231200\left(x^2-864x^3\right)y^4\\ &+332190720x^3y^5\Big)\end{aligned}$$

For $g=3$, the corresponding recursion relation for $\mathtt{F}_3(q,p)$ follows directly from [BCOV2] (, see also [KKV]). We can also find the rational function $f_3(x,y)$ of the form stated above, although we do not reproduce its lengthy form here.

Remark. (1) $f_g(x,y)$ is the holomorphic ambiguity in the solutions of BCOV holomorphic anomaly equation. As clear from (4.4) and (4.6), $w_0(x)^{2g-2}f_g(q,p)$ does not behave as a modular form under $t_1\to(at_1+b)/(ct_1+d)$. This means that the holomorphic ambiguity in the solutions of BCOV equation differs from the ambiguity $\sum_n f_{2g+6n-2}(E_4(q),E_6(q))p^n$ which arises when solving the holomorphic anomaly equation (1.4).
(2) It is worth while writing here the form of the propagater $\mathtt{S}^{t_it_j}$ in the coordinate x,y, i.e. that defined by $\mathtt{S}^{t_it_j}=w_0(x)^2\sum_{k,l}\mathtt{S}^{x_kx_l}\frac{\partial t_i}{\partial x_k}\frac{\partial t_j}{\partial x_l}$. After some calculation, it is easy to derive $\mathtt{S}^{xx}=\mathtt{S}^{xy}=0$ and

$$\mathtt{S}^{yy} = 1\mathtt{K}_{yyy}\left(-\Gamma^{\,y}_{y\,y}-1y\right) \quad ,\quad \Gamma^{\,y}_{y\,y}=\partial y\partial t_2\partial\ \partial y\left(\partial t_2\partial y\right) \quad ,$$

where $\mathtt{S}^{t_2t_2}=w_0^2\ \mathtt{S}^{yy}(\partial t_2\partial y)^2$ and $\mathtt{K}_{yyy}=w_0(x)^2(\frac{dt_2}{dy})^3\mathtt{K}_{t_2t_2t_2}$.

4.3 $\mathbf{P}^1 \times \mathbf{P}^1$. As a slightly different two parameter model, we may consider a local limit to a smooth divisor $\mathbf{P}^1 \times \mathbf{P}^1$ in a Calabi-Yau 3-fold. A Calabi-Yau model containing this surface may be realized as an elliptic fibration over $\mathbf{P}^1 \times \mathbf{P}^1$. The local mirror limit is a limit in which the volume of the fiber goes to infinity. And the resulting space may be identified as a non-compact Calabi-Yau manifold, $K_{\mathbf{P}^1\times\mathbf{P}^1} \to \mathbf{P}^1 \times \mathbf{P}^1$. Then the cohomology classes of compact support my be identified with those of the base space $\mathbf{P}^1 \times \mathbf{P}^1$. For a positive basis of $H^2(\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{Z})$, we choose the hyperplane classes H_1 and H_2 from each $\mathbf{P}^1$. Then under the local mirror symmetry limit, we have the generating function for Gromov-Witten invariants of $\mathbf{P}^1 \times \mathbf{P}^1$ which we parametrize by

$$\mathtt{F}_g(q,p) = \sum_{\beta\in H^2(\mathbf{P}^1\times\mathbf{P}^1,\mathbf{Z})} N_g(\beta) q^{(\beta,H_2)} p^{(\beta,H_1+H_2)} \quad .$$

Where a special parametrization for q, p has been chosen so that we can utilize the Segre embedding, $\mathbf{P}^1 \times \mathbf{P}^1$ into $\mathbf{P}^3$ as degree 2 surface. Namely, the diagonal direction $H_1 + H_2$ may be identified with the class coming from the hyperplane class of $\mathbf{P}^3$. The reduction of BCOV holomorphic anomaly equation to the diagonal one parameter subspace ($q = 1$) has been studied in [KKV], and our parametrization naturally recovers two parameters, H_2 and $H_1 + H_2$, from this one parameter reduction. Since the calculations are parallel to those appeared in $\frac{1}{2}K3$ case, here we simply write corresponding formulas for $\mathtt{F}_g(q,p)$.

The hypergeometric series we start with is given by [1]

$$w_0(x,y) = \sum_{n,m\geq 0} c(n,m) x^n y^m, \;\; c(n,m) = \frac{1}{\Gamma(1-n+m)^2\Gamma(1+n)\Gamma(1+n+2m)}.$$

As before, the mirror map is defined by $2\pi\sqrt{-1}t_i = \frac{\partial_{\rho_i} w_0(x,y,\rho_1,\rho_2)}{w_0(x,y)}|_{\rho_i=0}$. Then again we find a lower triangular form for $x = x(q,p), y = y(q,p)$ as

$$\begin{aligned} x &= q \\ y &= p - (2+2q)p^2 + (3+3q^2)p^3 - (4+4q+4q^2+4q^3)p^4 + O(p^5) \; . \end{aligned}$$

By using mirror symmetry, we can write $\mathtt{F}_0(q,p)$ in terms of hypergeometric series $w_0(x,y) = 1, \partial_{\rho_1} w_0(x,y), \partial_{\rho_2} w_0(x,y)$ and $\partial_{\rho_1}\partial_{\rho_2} w_0(x,y)$. The genus one function and the propagater has similar form as before;

$$\begin{aligned} \mathtt{F}_1(q,p) &= \frac{1}{2}\log\left\{ \big(1+16y^2(1-x)^2 - 8y(1+x)\big)^{-16} y^{-76} \partial y \partial t_2 \right\} \quad , \\ \mathtt{S}^{t_2 t_2} &= -1 \mathtt{K}_{t_2t_2t_2} \partial\, \partial t_2 \log\left(y \partial y \partial t_2\right) \quad , \end{aligned}$$

with $\mathtt{K}_{t_2t_2t_2} = \partial_{t_2}\partial_{t_2}\partial_{t_2}\mathtt{F}_0(q,p) = -1 - (2+2q)p - (2+32q+2q^2)p^2 + \cdots$. When we write the propagater in x, y coordinate we have

$$\mathtt{S}^{yy} = 1\mathtt{K}_{yyy}\left(-\Gamma^{\,y}_{y\,y} - 1y\right) \quad , \quad \Gamma^{\,y}_{y\,y} = \partial y \partial t_2 \partial\, \partial y \left(\partial t_2 \partial y\right) \quad ,$$

where $\mathtt{S}^{t_2t_2} = w_0^2\, \mathtt{S}^{yy} (\partial t_2 \partial y)^2$ and $\mathtt{K}_{yyy} = w_0(x)^2 (\frac{dt_2}{dy})^3 \mathtt{K}_{t_2t_2t_2}$.

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[1] This corresponds to the choice of the "charge vector" $l^{(1)} = (0;0,0,-1,1,-1,1,0), l^{(2)} = (0;0,0,1,0,1,0,2,0), l^{(3)} = (-6;3,2,0,0,0,0,1)$ for the elliptic Calabi-Yau threefold.

Now BCOV recursion formula for $\mathtt{F}_2$ is the same as the previous case (4.2), and for the holomorphic ambiguity $f_2(x,y)$ we find

$$f_2 = \Big(-11y(1+x) + 12y^2(31+58x+31x^2) - 16y^3(333+595x+595x^2+333x^3) \\ + 64y^4(1-x)(551+443x-443x^2-551x^3) - 107520y^5(1-x)^4(1+x) \\ + 122880y^6(1-x)^6\Big) \Big/ \Big(720\left(1+16y^2(1-x)^2-8y(1+x)\right)^2\Big) \quad .$$

Here the holomorphic ambiguity $f_2(x,y)$ has been fixed by requiring the vanishing for BPS numbers $n_2(aH_1+bH_2)$ for lower degrees a,b, and one known result $n(4H_1+2H_2)=116$ in [KKV].

In the following tables, we have listed the BPS numbers $n_g(a,b)=n_g(aH_1+bH_2)$ up to genus two, which result from the Gopakumar-Vafa formula (1.5).

a \ b	0	1	2	3	4	5	6	7
0	0	-2	0	0	0	0	0	0
1	-2	-4	-6	-8	-10	-12	-14	-16
2	0	-6	-32	-110	-288	-644	-1280	-2340
3	0	-8	-110	-756	-3556	-13072	-40338	-109120
4	0	-10	-288	-3556	-27264	-153324	-690400	-2627482
5	0	-12	-644	-13072	-153324	-1252040	-7877210	-40635264
6	0	-14	-1280	-40338	-690400	-7877210	-67008672	-455426686
7	0	-16	-2340	-109120	-2627482	-40635264	-455426686	-3986927140

Table 6. Genus zero BPS numbers $n_0(a,b)=n_0(aH_1+bH_2)$.

a \ b	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	9	68	300	988	2698	6444
3	0	0	68	1016	7792	41376	172124	599856
4	0	0	300	7792	95313	760764	4552692	22056772
5	0	0	988	41376	760764	8695048	71859628	467274816
6	0	0	2698	172124	4552692	71859628	795165949	6755756732
7	0	0	6444	599856	22056772	467274816	6755756732	73400088512

Table 7. Genus one BPS numbers $n_1(a,b)=n_1(aH_1+bH_2)$.

a \ b	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	-12	-116	-628	-2488	-8036
3	0	0	-12	-580	-8042	-64624	-371980	-1697704
4	0	0	-116	-8042	-167936	-1964440	-15913228	-99308018
5	0	0	-628	-64624	-1964440	-32242268	-355307838	-2940850912
6	0	0	-2488	-371980	-15913228	-355307838	-5182075136	-55512436778
7	0	0	-8036	-1697704	-99308018	-2940850912	-55512436778	-754509553664

Table 8. Genus two BPS numbers $n_2(a,b)=n_2(aH_1+bH_2)$.

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